Distinguished Arthur parameters and relative discrete series

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Abstract

Let F be a nonarchimedean field of characteristic zero. Let $\mathbf{G}$ be a connected reductive group that is split over F. Let $\mathbf{X}$ be a homogeneous spherical $\mathbf{G}$-variety with F-points $X = \mathbf{X}(F)$. From the point of view of harmonic analysis, it is interesting to ask: when does an admissible representation π of $G = \mathbf{G}(F)$ occur in $L^2(X)$? Sakellaridis and Venkatesh (Astérisque 396) have conjectured that π occurs in $L^2(X)$ if and only if π appears in an Arthur packet attached to a "distinguished" Arthur parameter $\psi : \mathcal{L}_F \times \mathrm{SL}(2,\mathbb{C}) \to G^\vee$, that is, ψ factors through a "distinguished morphism" $\varrho : G_X^\vee \times \mathrm{SL}(2,\mathbb{C}) \to G^\vee$, where $G_X^\vee$ is a complex dual group associated to $\mathbf{X}$, via a tempered Langlands parameter $\phi : \mathcal{L}_F \to G_X^\vee$. We will discuss several instances of a refinement of this conjecture for representations in the discrete spectrum of X in the case that $\mathbf{G} = \mathbf{GL}_N$ and $\mathbf{X}$ is a p-adic symmetric space.

Road map

- Distinguished Arthur parameters
  - The dual group $G_X^\vee$ and distinguished morphisms
  - Local conjectures of Sakellaridis and Venkatesh [SV17]
- Example 1: Linear periods and functorial transfer from $\mathrm{SO}(2n+1)$
- Example 2: Symplectic periods and Speh representations
- Example 3*: Galois distinction and unitary groups
- Epilogue: It seems everyone is working on $\mathbf{G}_2$ these days...

Groups, fields and varieties

- F: a nonarchimedean local field, $\mathrm{char}(F) = 0$
- $\mathcal{L}_F = \mathcal{W}_F \times \mathrm{SL}(2, \mathbb{C})$: local Langlands group
- $\mathbf{G}$: a connected reductive split group over F
- $\mathbf{X}$: a homogeneous spherical $\mathbf{G}$-variety
  - $\mathbf{X}$ is a normal F-variety with a transitive $\mathbf{G}$ action
  - spherical: a Borel subgroup of $\mathbf{G}$ has a Zariski-dense orbit $\mathbf{X}^\dagger$
  - Fix $x_0 \in X^\dagger$, $\mathbf{H} = \mathrm{Stab}_{\mathbf{G}}(x_0)$
  - $\mathbf{X} \cong \mathbf{H}\backslash\mathbf{G}$
- In the Examples: $\mathbf{H} = \mathbf{G}^\theta$, $\theta \in \mathrm{Aut}_F(\mathbf{G})$ an involution
  - $\mathbf{X} = \mathbf{G}^\theta\backslash\mathbf{G}$ a symmetric variety
  - J.S. Milne, *Algebraic groups*, (2017), Theorem 5.28 / Chapter 7e.
- Denote F-points by $G = \mathbf{G}(F)$, $X = \mathbf{X}(F)$, etc.
- All representations are smooth, admissible, on $\mathbb{C}$ vector spaces

Harmonic analysis on X: distinguished representations

- Goal: understand $L^2(X)$ as a unitary representation of G
  - discrete spectrum?
  - Plancherel measure and direct integral decomposition?
  - support of Plancherel measure?
- an rep. (π, V) occurs in $C^\infty(X)$ if an only if $\mathrm{Hom}_H(\pi, 1) \neq \{0\}$
- $\lambda \in \mathrm{Hom}_H(\pi, 1)$ nonzero, $\forall\, v \in V \mapsto \varphi_{\lambda,v} \in C^\infty(X)$

$$\varphi_{\lambda,v}(g) = \langle \lambda, \pi(g)v \rangle$$

$v \mapsto \varphi_{\lambda,v}$ is a G-morphism

Definition

(π, V) is *H-distinguished* if $\mathrm{Hom}_H(\pi, 1) \neq \{0\}$

Definition

An irred. (unitary) H-dist. rep. (π, V) is a relative discrete series (RDS) if (π, V) is isomorphic to a direct summand of $L^2(X)$

The dual group $G_X^\vee$

- $G^\vee$ the complex dual group of **G**
- Following work of Gaitsgory and Nadler, Sakellaridis and Venkatesh have associated to **X** a complex reductive group $G_X^\vee$ which we'll call the dual group of **X**
  - $\mathbf{X} \rightsquigarrow \Sigma_X$: "spherical roots" of $\mathbf{A}_X \cong \mathbf{A}_0/(\mathbf{A}_0 \cap \mathbf{H})$
  - W_X little Weyl Group
  - Δ_X "simple normalized spherical roots"; $\Phi_X = W_X \cdot \Delta_X$
  - (Φ_X, Δ_X) based root system; $R = \mathrm{Hom}_F(A_X, \mathbb{G}_m)$
  - $(R^\vee, \Phi_X^\vee, \Delta_X^\vee, R, \Phi_X, \Delta_X)$ is a based root datum
- $G_X^\vee$ is the complex algebraic group associated to this root datum
  - canonical maximal torus A_X^* and map $A_X^* \to A^*$ dual to $\mathbf{A}_0 \to \mathbf{A}_X$
  - $A^* \subset G^\vee$ the complex dual torus of $\mathbf{A}_0$

Question

Can we extend the canonical map $A_X^* \to A^*$ to a morphism $G_X^\vee \to G^\vee$?

The dual group $G_X^\vee$

Example

$\mathbf{X}$	$G_X^\vee$
$(\mathbf{GL}_n \times \mathbf{GL}_n)\backslash\mathbf{GL}_{2n}$	$\mathbf{Sp}(2n, \mathbb{C})$
$\mathbf{Sp}_{2n}\backslash\mathbf{GL}_{2n}$	$\mathbf{GL}(n, \mathbb{C})$
GGP case[1]: $\mathbf{SO}_{n-1}\backslash\mathbf{SO}_n$	$\widetilde{\mathrm{SL}}_2$
$\mathbf{SL}_3\backslash\mathbf{G}_2$	$\widetilde{\mathrm{SL}}_2$
$\mathbf{G}_2\backslash Spin_7$	$\mathrm{SL}(2, \mathbb{C})$

[1] This is the setting of Gan–Gross–Prasad.

Distinguished morphisms

Definition

A distinguished morphism $\varrho : G_X^\vee \times \mathrm{SL}(2,\mathbb{C}) \to G^\vee$ is a group homomorphism such that

1. the restriction of ϱ to $G_X^\vee$ extends the canonical map of tori $A_X^* \to A^*$
2. for every simple normalized spherical root $\gamma_0 \in \Delta_X$, the corresponding root space of the Lie algebra $\mathfrak{g}_X^\vee$ maps into the sum of root spaces of its associated roots under the differential of ϱ
3. the restriction of ϱ to the $\mathrm{SL}(2,\mathbb{C})$ factor is a principal morphism into $M_0^\vee \subset G^\vee$ with weight $2\rho_{M_0} : \mathbb{G}_m \to G^\vee$, where $\mathbb{G}_m$ is identified with the maximal torus of $\mathrm{SL}(2,\mathbb{C})$ via $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $2\rho_{M_0}$ is the sum of the positive roots of $\mathbf{A}_0$ in $\mathbf{M}_0$.

- Knop–Schalke [KS17]: Distinguished morphisms exist
- S–V: Distinguished morphisms are unique up to A^*-conjugacy [SV17, Proposition 3.4.3]

Distinguished morphisms: examples

Example

- $\mathbf{X} = (\mathbf{GL}_n \times \mathbf{GL}_n)\backslash\mathbf{GL}_{2n}$
- $G_X^\vee = \mathbf{Sp}(2n, \mathbb{C})$ and $G^\vee = \mathbf{GL}(2n, \mathbb{C})$
- The distinguished morphism $\varrho : G_X^\vee \times \mathrm{SL}(2, \mathbb{C}) \to G^\vee$ is trivial on the $\mathrm{SL}(2, \mathbb{C})$ factor and is given by the inclusion map $G_X^\vee \hookrightarrow G^\vee$.

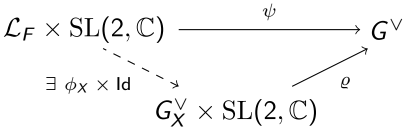
Example (SV, Example 1.3.2)

- $\mathbf{X} = \mathbf{Sp}_{2n}\backslash\mathbf{GL}_{2n}$
- $G_X^\vee = \mathbf{GL}(n, \mathbb{C})$ and $G^\vee = \mathbf{GL}(2n, \mathbb{C})$
- The distinguished morphism $\varrho : \mathbf{GL}(n, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \mathbf{GL}(2n, \mathbb{C})$ is given by the tensor product of the standard n-dimensional representation of $\mathbf{GL}(n, \mathbb{C})$ with the standard 2-dimensional representation $\mathcal{S}(2)$ of $\mathrm{SL}(2, \mathbb{C})$.

Sakellaridis & Venkatesh: Local conjectures I

Definition

An A-parameter $\psi : \mathcal{L}_F \times \mathrm{SL}(2,\mathbb{C}) \to G^\vee$ is X-distinguished if it factors through the distinguished morphism $\varrho : G_X^\vee \times \mathrm{SL}(2,\mathbb{C}) \to G^\vee$.



That is, ψ is X-distinguished if and only if there exists a tempered (bounded on $\mathcal{W}_F$) L-parameter $\phi_X : \mathcal{L}_F \to G_X^\vee$ such that $\psi = \varrho \circ (\phi_X \times \mathrm{Id})$.

Sakellaridis & Venkatesh: Local conjectures II

We recall the following conjecture [SV17, Conjectures 1.3.1 and 16.2.2].

Conjecture (Sakellaridis–Venkatesh)

The support of the Plancherel measure for $L^2(X)$, as a representation of G, is contained in the union of Arthur packets attached to X-distinguished A-parameters.

Definition

An X-distinguished A-parameter is X-elliptic if it factors through ϱ via an elliptic L-parameter $\phi_X : \mathcal{L}_F \to G_X^\vee$, that is, the image of ϕ_X is not contained in any proper Levi subgroup of $G_X^\vee$.

The following is part of [SV17, Conjecture 16.2.2].

Conjecture (Sakellaridis–Venkatesh)

A relative discrete series representation π in $L^2(X)$ is contained in an Arthur packet corresponding to an X-distinguished X-elliptic A-parameter.

$(\mathbf{GL}_n \times \mathbf{GL}_n)\backslash\mathbf{GL}_{2n}$

- $\mathbf{G} = \mathbf{GL}_{2n}$ with $n \geq 2$
- $\mathbf{H} = \mathbf{GL}_n \times \mathbf{GL}_n$ is the fixed points of $\theta = \operatorname{Int}\operatorname{diag}(I_n, -I_n)$.
- nonzero $\lambda \in \operatorname{Hom}_H(\pi, 1)$ referred to as a (local) linear period

Theorem (Jacquet–Rallis [JR96])

Let (π, V) be an irreducible admissible representation of G.

1. $\dim(\operatorname{Hom}_H(\pi, 1)) \leq 1$.
2. *If* $\dim \operatorname{Hom}_H(\pi, 1) = 1$, *then* $\widetilde{\pi} \cong \pi$.

Theorem (Matringe [Mat14, Proposition 6.1])

Suppose that π is a square integrable representation of G, then π is H-distinguished if and only if the exterior square L-function $L(s, \pi, \wedge^2)$ has a pole at $s = 0$.

RDS for $(\mathbf{GL}_n \times \mathbf{GL}_n)\backslash\mathbf{GL}_{2n}$

Suppose that F has odd residual characteristic.
$G_m = \mathbf{GL}_m(F)$ and $H_m = \mathbf{GL}_{m/2}(F) \times \mathbf{GL}_{m/2}(F)$

Theorem (S. 2017)

Let $\{\delta_i\}_{i=1}^d$ be pairwise inequivalent H_{m_i}-distinguished discrete series representations of G_{m_i}. The parabolically induced representation $\pi = \delta_1 \times \ldots \times \delta_d$ is a relative discrete series representation.

Theorem (Matringe [Mat14, Theorem 6.1])

Suppose that $m = kr$ is even. Let ρ be an irreducible supercuspidal representation of G_r. Let $\pi = \mathrm{St}(k, \rho)$ be a generalized Steinberg representation of G_m.

1. *If k is odd, then r must be even, and π is H_m-distinguished if and only if $L(s, \rho, \wedge^2)$ has a pole at $s = 0$ if and only if ρ is H_r-distinguished*
2. *If k is even, then π is H_m-distinguished if and only if $L(s, \rho, \mathrm{Sym}^2)$ has a pole at $s = 0$.*

Distinguished parameters for $(\mathbf{GL}_n \times \mathbf{GL}_n)\backslash\mathbf{GL}_{2n}$ I

- $G_X^\vee = \mathbf{Sp}(2n, \mathbb{C})$ and $G^\vee = \mathbf{GL}(2n, \mathbb{C})$
- The distinguished morphism $\varrho : G_X^\vee \times \mathrm{SL}(2, \mathbb{C}) \to G^\vee$ is trivial on the $\mathrm{SL}(2, \mathbb{C})$ factor and is given by the inclusion map $G_X^\vee \hookrightarrow G^\vee$.
- An X-distinguished A-parameter is

$$\phi \otimes 1 : \mathcal{L}_F \times \mathrm{SL}(2, \mathbb{C}) \to G^\vee$$

  where ϕ is a tempered symplectic L-parameter for G.
- Thus S–V predicts that the RDS for X are tempered representations
  - Now known by work of Beuzart-Plessis [BP18a], and Gurevich and Offen [GO16]
- The L-parameter $\phi_\delta : \mathcal{L}_F \to \mathbf{GL}(m, \mathbb{C})$ of the generalized Steinberg representation $\delta = \mathrm{St}(k, \rho)$ is equal to $\phi_\delta = \phi_\rho \otimes \mathcal{S}(k)$,

Distinguished parameters for $(\mathbf{GL}_n \times \mathbf{GL}_n)\backslash\mathbf{GL}_{2n}$ II

- The following proposition is a consequence of [JNQ08, Theorem 5.5] and Matringe's theorem

Proposition

Suppose $m = kr$ is even. Let ρ be an irreducible self-contragredient supercuspidal representation of G_r. If $\delta = \mathrm{St}(k, \rho)$ is H_m-distinguished, then the image of the L-parameter ϕ_δ is contained in $\mathbf{Sp}(m, \mathbb{C})$. □

Theorem (S. [Smi18a])

Let π be a known relative discrete series for $(\mathbf{GL}_n \times \mathbf{GL}_n)\backslash\mathbf{GL}_{2n}$. The A-parameter $\phi_\pi \otimes 1$ of π is X-distinguished and X-discrete. □

- In fact, the known RDS are the only tempered H-distinguished representations of G with X-distinguished and X-discrete parameters... so we expect that these are all of the RDS.

Plancherel formula for $(\mathbf{GL}_n \times \mathbf{GL}_n)\backslash\mathbf{GL}_{2n}$

- The Plancherel formula for $H\backslash G$ has been obtained by N. Duhamel [December 2019, arXiv:1912.08497]
- This also gives the Plancherel formula for the Shalika model of G

Theorem (Duhamel)

There exists a G-equivariant isomorphism of unitary representations

$$L^2((\mathbf{GL}_n(F) \times \mathbf{GL}_n(F))\backslash\mathbf{GL}_{2n}(F)) \cong \int^{\oplus}_{\Pi_{\mathrm{t}}(\mathbf{SO}_{2n+1}(F))} T(\pi) d\mu(\pi)$$

where $d\mu$ is the Plancherel measure on $\Pi_{\mathrm{t}}(\mathbf{SO}_{2n+1}(F))$, and $T : \widetilde{\Pi}_{\mathrm{t}}(\mathbf{SO}_{2n+1}(F)) \to \Pi_{\mathrm{t}}(\mathbf{GL}_{2n}(F))$ is the local Langlands functorial transfer from tempered L-packets $\widetilde{\Pi}_{\mathrm{t}}(\mathbf{SO}_{2n+1}(F))$. □

- Thus the Conjectures of Sakellaridis and Venkatesh hold for $(\mathbf{GL}_n \times \mathbf{GL}_n)\backslash\mathbf{GL}_{2n}$ (up to verifying the exhaustion of the discrete spectrum, which is expected).

$\mathbf{Sp}_{2n}\backslash\mathbf{GL}_{2n}$ and Speh representations

- $\mathbf{G} = \mathbf{GL}_{2n}$ with $n \geq 2$
- $\mathbf{H} = \mathbf{Sp}_{2n}$ is the fixed points of $\theta(g) = J^{-1}\,{}^t g^{-1} J$ where

$$J = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$$

- nonzero $\lambda \in \mathrm{Hom}_H(\pi, 1)$ referred to as a (local) symplectic period
- H-dist. unitary representations are classified by [OS07, OS08a]

Let δ be a discrete series rep. of G_n.

$$0 \to \mathcal{Z}(\delta, 2) \to \nu^{1/2}\delta \times \nu^{-1/2}\delta \to \mathcal{U}(\delta, 2) \to 0$$

- $\mathcal{Z}(\delta, 2)$ unique irreducible generic subrep.
- $\mathcal{U}(\delta, 2)$ unique irred. non-tempered quotient – Speh representation

Disjointness of models

Heumos and Rallis [HR90]

- $\mathcal{U}(\delta, 2)$ is H-distinguished
- generic representations of G cannot be H-distinguished
- if π is an irred. adm. rep of G, then

$$\dim(\mathrm{Hom}_H(\pi, 1)) = \dim(\mathrm{Hom}_H(\widetilde{\pi}, 1)) \leq 1$$

- much more... "unitary disjointness of models" [HR90, Theorem 3.1]

Existence of mixed models for all irred. unitary π of $\mathbf{GL}_n(F)$ [OS08b]

RDS for $\mathbf{Sp}_{2n}\backslash\mathbf{GL}_{2n}$

The following theorem is an unpublished result of H. Jacquet.

Theorem (Jacquet, S. [Smi20])

Let δ be a discrete series representation of $\mathbf{GL}_n(F)$.
The Speh representation $\mathcal{U}(\delta, 2)$ of $\mathbf{GL}_{2n}(F)$ is relative discrete series. □

Remark

- $\mathcal{U}(\delta, 2)$ is a non-tempered representation of $\mathbf{GL}_{2n}(F)$
- but appears in the discrete spectrum of $\mathbf{Sp}_{2n}\backslash\mathbf{GL}_{2n}$
- no discrete series of $\mathbf{GL}_{2n}$ is $\mathbf{Sp}_{2n}$-discrete

Distinguished parameters for $\mathbf{Sp}_{2n}\backslash\mathbf{GL}_{2n}$ I

- $G_X^\vee = \mathbf{GL}(n,\mathbb{C})$ and $G^\vee = \mathbf{GL}(2n,\mathbb{C})$
- The distinguished morphism $\varrho : \mathbf{GL}(n,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathbf{GL}(2n,\mathbb{C})$ is given by the tensor product of the standard n-dimensional representation of $\mathbf{GL}(n,\mathbb{C})$ with the standard 2-dimensional representation $\mathcal{S}(2)$ of $\mathrm{SL}(2,\mathbb{C})$.

Proposition (S. on Shoulders of Giants)

Let π be an irreducible unitary $\mathbf{Sp}_{2n}(F)$-distinguished representations of $\mathbf{GL}_{2n}(F)$. Let $\psi_\pi : \mathcal{L}_F \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{C})$ be the A-parameter of π. The A-parameter ψ_π is X-distinguished and X-elliptic if and only if π is isomorphic to a Speh representation $\mathcal{U}(\delta, 2)$ for some discrete series representation δ of $\mathbf{GL}_n(F)$. □

Distinguished parameters for $\mathbf{Sp}_{2n}\backslash\mathbf{GL}_{2n}$ II

Sketch of the proof.

- $\psi : \mathcal{L}_F \times \mathrm{SL}(2,\mathbb{C}) \to \mathbf{GL}(2n,\mathbb{C})$ be the A-parameter of π
- ψ is X-distinguished $\iff \psi = \phi_X \otimes \mathcal{S}(2)$, where $\phi_X : \mathcal{L}_F \to \mathrm{GL}(n,\mathbb{C})$ is a tempered L-parameter
- ψ is X-elliptic $\iff$ is ϕ_X is elliptic in $\mathbf{GL}(n,\mathbb{C})$
- ψ is X-dist and X-ell $\iff \psi = \phi_\delta \otimes \mathcal{S}(2)$ where ϕ_δ is the L-parameter of a $\mathbf{GL}_n(F)$ discrete series $\delta \iff \pi \cong \mathcal{U}(\delta, 2)$

This all relies on:

- Tadić's classification of the unitary dual of $\mathbf{GL}_n$
- Offen and Sayag's classification of $\mathbf{Sp}_{2n}(F)$-distinguished reps.
- The Local Langlands Correspondence for $\mathbf{GL}_n$ (i.e., "Giants")

□

Non-split examples: refined conjectures of D. Prasad

- S–V consider only the case when $\mathbf{G}$ is split over F.
- D. Prasad [Pra15] has refined the S–V conjectures for $\mathrm{Res}_{E/F}\mathbf{GL}_n$ where E/F is a quadratic Galois extension
- Prasad emphasizes the "geometry of L-parameters" à la A–B–V

Example

- The space $\mathbf{GL}_n(F)\backslash\mathbf{GL}_n(E)$ has been extensively studied.
- See [Pra15, Conjecture 2] for the analogue of S–V.
- Now "mostly" resolved by [BP18a, BP18b], including an explicit Plancherel formula in terms of stable/unstable base change from quasi-split unitary groups and an explicit description of the entire discrete spectrum (*cf.* [Smi18c]).

$\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_{2n}(E)$

- $G = \mathrm{Res}_{E/F}\mathbf{GL}_n(F) \simeq \mathbf{GL}_n(E)$
- $H = \mathbf{U}_{E/F}(F)$ a quasi-split unitary group
- Beuzart-Plessis extends the distinguished morphism to be compatible with quadratic base-change
- The following theorem is a special case of Beuzart-Plessis's result

Theorem (Beuzart-Plessis [BP20])

There exists a G-equivariant isomorphism of unitary representations

$$L^2(H\backslash G) \cong \int_{\Pi_{\mathrm{t}}(\mathbf{GL}_{2n}(F))}^{\oplus} \mathrm{bc}(\pi)\,d\mu(\pi)$$

where $d\mu$ is the Plancherel measure on $\Pi_{\mathrm{t}}(\mathbf{GL}_{2n}(F))$, and $\mathrm{bc} : \Pi(\mathbf{GL}_{2n}(F)) \to \Pi(\mathbf{GL}_{2n}(E))$ is Arthur–Clozel's [AC89] quadratic base-change. □

Results for $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_{2n}(E)$

- $\sigma \in \mathrm{Gal}(E/F)$ nontrivial
- $\Pi^\sigma(G) = \{\pi \in \Pi(G) : \pi \cong {}^\sigma\pi\}$
- $\eta : F^\times \to \mathbb{C}^\times$ quadratic character from LCFT
- $\mathrm{bc} : \Pi(\mathbf{GL}_n(F)) \twoheadrightarrow \Pi^\sigma(\mathbf{GL}_n(E))$ and $\mathrm{bc}(\pi') = \mathrm{bc}(\pi' \otimes \eta)$

The following is a special case of [BP20, Corollary 6.1.1]

Theorem

A RDS π for $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_{2n}(E)$ is either

1. *An H-dist. discrete series of G, i.e., $\pi = \mathrm{bc}(\pi')$ where $\pi' \not\cong \pi' \otimes \eta$*
2. *A non-discrete series (but tempered) rep. $\pi = \mathrm{bc}(\pi') \cong \tau \times {}^\sigma\tau$, where $\pi' \cong \pi' \otimes \eta$, and $\tau \not\cong {}^\sigma\tau$ is a discrete series of $\mathbf{GL}_n(E)$.*

□

Remark

Previously, in [Smi18b, Theorem 5.11] it was shown directly that the representations in (2) are RDS; however, exhaustion of the discrete spectrum was not then known.

Gan and Gomez: Low-rank spherical varieties

Example

X	$G_X^\vee$
$\mathbf{G}_2\backslash Spin_7$	$\mathrm{SL}(2,\mathbb{C})$
$\mathbf{G}_2\backslash Spin_8$	$\mathrm{SL}(2,\mathbb{C})^3/\Delta\mu_2$
$\mathbf{SL}_3\backslash\mathbf{G}_2$	$\widetilde{\mathrm{SL}}_2$

For these $\mathbf{G}_2$ cases, and many other classical and exceptional instances of low-rank spherical varieties, Gan and Gomez [GG14] have proven the "support of the Plancherel measure" conjecture of Sakellaridis and Venkatesh.

Question

What about the relative discrete series?

Remark

We are in the very earliest stages of a new project (with S. Dijols) to consider the symmetric space $\mathbf{SO}(4)\backslash\mathbf{G}_2$

Thank you!

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