Support of closed orbit relative matrix coefficients

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October 11, 2018

Outline

1. p-adic symmetric spaces and distinguished representations
2. equivariant linear forms on Jacquet modules
3. characterizing (H, χ, λ)-relative supercuspidality
4. an (H, χ)-relative subrepresentation theorem
5. support of closed orbit relative matrix coefficients: representations that are not relatively supercuspidal

Remark

- The results in items 2-4 generalize the work of Kato and Takano (2008) and have recently been obtained independently by Takeda.
- Delorme (2010) obtained similar results to those in 2 and 3 following the methods of Lagier (2008)

Notation

- F a nonarchimedean local field with odd residual characteristic
- $\mathbf{G}$ connected reductive F-group; with F-points G
- θ an F-rational involution
- $\mathbf{H} = \mathbf{G}^\theta$ θ-fixed points; F-points H
- $X = G/H$ is a p-adic symmetric space

All representations are smooth and on complex vector spaces.

Definition

Let χ be a quasi-character of H. A representation (π, V) of G is (H, χ)-distinguished if there is a nonzero element λ in $\mathrm{Hom}_H(\pi, \chi)$.

- If $\chi = 1$, then we say that π is H-distinguished
- If π has central character ω, then π is an ω-representation
- If π is an (H, χ)-distinguished ω-representation, then $\chi|_{H \cap Z_G} = \omega$

Relative matrix coefficients

(π, V) an (H,χ)-distinguished ω-representation, fix $\lambda \neq 0 \in \mathrm{Hom}_H(\pi,\chi)$

- $\forall v \in V$, define the λ-relative matrix coefficient $\varphi_{\lambda,v} : G \to \mathbb{C}$ by

$$\varphi_{\lambda,v}(g) = \langle \lambda, \pi(g^{-1})v \rangle$$

- π smooth ω-rep $\Rightarrow \varphi_{\lambda,v} \in C^\infty_\omega(G) \subset C^\infty(G)$, where

$$C^\infty_\omega(G) = \{f \in C^\infty(G) : f(zg) = \omega(z^{-1})f(g), \forall z \in Z_G, g \in G\}$$

- The map $v \mapsto \varphi_{\lambda,v}$ intertwines (π, V) and the left-regular representation of G on

$$C^\infty(G,H,\chi) = \{f \in C^\infty(G) : f(gh) = \chi(h^{-1})f(g)\} \cong \mathrm{Ind}_H^G \chi$$

- since π is an ω-rep, $\varphi_{\lambda,v} \in C^\infty_\omega(G,H,\chi) = C^\infty(G,H,\chi) \cap C^\infty_\omega(G)$

Support modulo $Z_G H$

- Observe: $\forall g \in G, z \in Z_G, h \in H$

$$\varphi_{\lambda,v}(gzh) = \langle \lambda, \pi(h^{-1}z^{-1}g^{-1})v \rangle = \chi(h^{-1})\omega(z^{-1})\varphi_{\lambda,v}(g)$$

  and it makes sense to consider the support of $\varphi_{\lambda,v}$ modulo $Z_G H$.
- Define $C^\infty_{\omega,0}(G,H,\chi)$ to be the space

$$\{f \in C^\infty_\omega(G,H,\chi) : \text{Supp}(f) \text{ has compact image in } G/Z_G H\}$$

Definition

The (H,χ)-distinguished ω-representation (π, V) is said to be:

1. (H,χ,λ)-relatively supercuspidal iff $\varphi_{\lambda,v} \in C^\infty_{\omega,0}(G,H,\chi)$, $\forall v \in V$
2. (H,χ)-relatively supercuspidal if and only if π is (H,χ,λ)-relatively supercuspidal for every $\lambda \in \text{Hom}_H(\pi,\chi)$

If $\chi = 1$ then we drop it from the notation

Invariant forms on Jacquet modules ($\chi = 1_H$)

A parabolic subgroup P of G is θ-split if $\theta(P)$ is opposite to P
If P is θ-split, then $M = P \cap \theta(P)$ is a θ-stable Levi factor of P

Theorem (Kato–Takano, Lagier; 2008)

Let (π, V) be an admissible H-distinguished representation of G. Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be nonzero. Let $P = MN$ be a θ-split parabolic subgroup of G with unipotent radical N and θ-stable Levi $M = P \cap \theta(P)$. There exists a linear functional $\lambda_N : V_N \to \mathbb{C}$, canonically associated to λ such that

1. $\lambda_N \in \mathrm{Hom}_{M^\theta}(\pi_N, 1)$ *is M^θ-invariant*
2. *The map $\mathrm{Hom}_H(\pi, 1) \to \mathrm{Hom}_{M^\theta}(\pi_N, 1)$ given by $\lambda \mapsto \lambda_N$ is linear*
3. *the map $\lambda \mapsto \lambda_N$ is compatible with the transitivity of Jacquet restriction*

Remark

λ_N is constructed from λ via Casselman's Canonical Lifting

Characterizing (H, λ)-relatively supercuspidal reps

Theorem (Kato–Takano, Lagier)

Let (π, V) be an admissible H-distinguished representation of G. Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be nonzero. Then (π, V) is (H, λ)-relatively supercuspidal if and only if $\lambda_N = 0$ for every proper θ-split parabolic subgroup $P = MN$ of G

Theorem (Kato–Takano)

Let (π, V) be an irreducible admissible H-distinguished representation of G. There exists a θ-split parabolic subgroup $P = MN$ of G and an irreducible M^θ-relatively supercuspidal representation (ρ, W) of M such that π is equivalent to a subrepresentation of the parabolically induced representation $\iota_P^G \rho$.

Goal

Generalize the work of Kato–Takano, Lagier to include (H, χ)-distinguished representations when χ is nontrivial

Equivariant linear forms on Jacquet modules (any χ)

Theorem (Delorme 2010, S., Takeda)

Let (π, V) be an admissible (H, χ)-distinguished representation of G. Let $\lambda \in \mathrm{Hom}_H(\pi, \chi)$ be nonzero. Let $P = MN$ be a θ-split parabolic subgroup of G. There exists a linear functional $\lambda_{N,\chi} : V_N \to \mathbb{C}$, canonically associated to λ such that

1. $\lambda_{N,\chi} \in \mathrm{Hom}_{M^\theta}(\pi_N, \chi|_{M^\theta})$
2. *The map* $\mathrm{Hom}_H(\pi, \chi) \to \mathrm{Hom}_{M^\theta}(\pi_N, \chi|_{M^\theta})$ *given by* $\lambda \mapsto \lambda_{N,\chi}$ *is linear*
3. *If $\chi = 1$, then $\lambda_{N,1} = \lambda_N$ is the form defined by Kato–Takano, Lagier*
4. $\lambda \mapsto \lambda_{N,\chi}$ *is compatible with the transitivity of Jacquet restriction*

Remark

- $\lambda_{N,\chi}$ is defined in exactly the same way as λ_N
- Proof: chase χ through the arguments of Kato–Takano, Lagier and use that χ is smooth to get inside $\ker \chi$ when taking Canonical Lifts

Characterizing (H, χ, λ)-relatively supercuspidal reps

Theorem (Delorme 2010, S., Takeda)

Let (π, V) be an admissible (H, χ)-distinguished representation of G. Let $\lambda \in \mathrm{Hom}_H(\pi, \chi)$ be nonzero. Then (π, V) is (H, χ, λ)-relatively supercuspidal if and only if $\lambda_{N,\chi} = 0$ for every proper θ-split parabolic subgroup $P = MN$ of G.

Theorem (S., Takeda)

Let (π, V) be an irreducible admissible (H, χ)-distinguished representation of G. There exists a θ-split parabolic subgroup $P = MN$ of G and an irreducible $(M^\theta, \chi|_{M^\theta})$-relatively supercuspidal representation (ρ, W) of M such that π is equivalent to a subrepresentation of the parabolically induced representation $\iota_P^G \rho$.

Remark

Takeda has also recently proved a subrepresentation theorem for H-relatively tempered representations

Representations that are **not** relatively supercuspidal

- $Q = LU$ a θ-stable parabolic subgroup with θ-stable Levi L
- μ a positive quasi-invariant measure on $Q^\theta \backslash H$
- ρ a smooth representation of L and let $\pi = \iota_Q^G \rho$

Lemma (Closed orbit equivariant forms)

The map $\lambda \mapsto \lambda^G$, *where*

$$\langle \lambda^G, \phi \rangle = \int_{Q^\theta \backslash H} \langle \lambda, \chi(h)^{-1} \phi(h) \rangle \; d\mu(h)$$

for $\phi \in V_\pi$, *is an injection of* $\mathrm{Hom}_{L^\theta}(\delta_Q^{1/2} \rho, \delta_{Q^\theta} \chi|_{L^\theta})$ *into* $\mathrm{Hom}_H(\pi, \chi)$.

- If $\delta_Q^{1/2}|_{L^\theta} = \delta_{Q^\theta}$, then $\mathrm{Hom}_{L^\theta}(\rho, \chi|_{L^\theta}) \hookrightarrow \mathrm{Hom}_H(\pi, \chi)$.

Question (Motivation)

Can we determine when $(\lambda^G)_{N,\chi}$ is nonzero by using the properties of λ?

A result on the support of functions on $G/Z_G H$

- Assume that $\chi = 1$ is the trivial character of H
- $Q = LU$ a θ-stable parabolic subgroup with θ-stable Levi L
- ρ an admissible (L^θ, χ')-distinguished representation of L, where $\chi' = \delta_{Q^\theta}\delta_Q^{-1/2}|_{L^\theta}$
- By assumption, $\exists$ nonzero $\lambda \in \mathrm{Hom}_{L^\theta}(\delta_Q^{1/2}\rho, \delta_{Q^\theta})$
- Define $\pi = \iota_Q^G \rho$ and build $\lambda^G \in \mathrm{Hom}_H(\pi, 1)$ from λ

Theorem (S.)

If π is (H, λ^G)-relatively supercuspidal, then ρ is $(L^\theta, \chi', \lambda)$-relatively supercuspidal.

Corollary

If $(\lambda^G)_N = 0$ for all proper θ-split parabolic subgroup $P = MN$ of G, then $\lambda_{N',\chi'} = 0$ for all proper θ-split parabolic subgroups $P' = M'N'$ of L.

A sketch of the proof

- Prove: If ρ is not $(L^\theta, \chi', \lambda)$-rsc, then $\pi = \iota_Q^G \rho$ is not (H, λ^G)-rsc
- Consider support of matrix coefficients modulo $S_G H$
- $\exists v \in V_\rho$ such that $\varphi_{\lambda,v}$ has non-compact support modulo $S_L L^\theta$
- Delorme–Sécherre and Benoist–Oh: Relative Cartan Decomposition: $L = \mathfrak{C}_L S^+ \mathfrak{X}_L^{-1} L^\theta$, $\mathfrak{X}_L \subset (\mathbf{L}^\theta C_{\mathbf{L}}(S))(F)$, S max (θ, F)-split torus of L
- WLOG: $\varphi_{\lambda,v}$ non-compactly support on S/S_L
- Let $K < G$ be compact open with Iwahori factorization wrt Q
- Define $f_v \in V_\pi$ to be zero off of $QK = Q(U^{\mathrm{op}} \cap K)$ and such that $f_v(\bar{u}) = v$ for $\bar{u} \in U^{\mathrm{op}} \cap K$

$$\varphi_{\lambda^G, f_v}(\ell) = \int_{Q^\theta \backslash H} \langle \lambda, \delta_Q^{1/2}(\ell^{-1}) \rho(\ell^{-1}) f_v(\ell h \ell^{-1}) \rangle \, d\mu(h) = c_\ell \cdot \delta_Q^{1/2}(\ell^{-1}) \varphi_{\lambda,v}(\ell)$$

$$\text{where } c_\ell = \int_{Q^\theta \backslash (\ell^{-1} K \ell)^\theta} d\mu(h) = \mu(Q^\theta \backslash (U^{\mathrm{op}} \cap \ell^{-1} K \ell)^\theta) > 0$$

Remark

We can prove the analogous result for nontrivial χ if we assume that $\chi : H \to \mathbb{R}_{>0}$ is positive valued

Theorem (S.)

Assume that $\chi : H \to \mathbb{R}_{>0}$ is positive valued. Assume that ρ is (L^θ, χ')-distinguished, where $\chi' = \delta_{Q^\theta}\delta_Q^{-1/2}\chi|_{L^\theta}$. If π is (H, χ, λ^G)-relatively supercuspidal, then ρ is $(L^\theta, \chi', \lambda)$-relatively supercuspidal.

Question (The open orbit)

If $P = MN$ is θ-split, τ is M^θ-distinguished, and the H-invariant form λ' on $\pi' = \iota_P^G\tau$ arises via the open orbit (i.e., via Blanc–Delorme) what can we say about $\lambda'_{N'}$ for θ-split $P' = M'N'$?

- This problem was addressed by J. Carmona and P. Delorme (Trans. Amer. Math. Soc. **366** (2014), no. 10, 5323–5377.)
- If τ is relatively supercuspidal and $P' \cap M \subsetneq M$, then $\lambda'_{N'} = 0$.

Thank you!