A-packets for quasisplit $GSp(2n)$ and $GSO(2n)$ over a p-adic field

Voganish Seminar

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1. Global motivations

k number field, $G = Sp(2n)$ over k

Thm (Arthur):

$$L^2_{disc}\left(G(k)\backslash G(\mathbb{A}_k)\right) = \bigoplus_{\psi \in \Psi_2(G)} \bigoplus_{\pi \in \Pi_\psi} m_\psi(\pi)\, \pi$$

$$1.\ \psi : L_k \times SL_2(\mathbb{C}) \longrightarrow \underset{SO(2n+1,\mathbb{C})}{\overset{\parallel}{G^\vee}} \xhookrightarrow{std} GL_{2n+1}(\mathbb{C})$$

Substituted by automorphic repns of $GL_{2n+1}(\mathbb{A}_k)$

2. $\Pi_\psi = \bigotimes_v{}' \Pi_{\psi_v} \qquad \Pi_{\psi_v} \subseteq \mathrm{Irr}(G(k_v)) \rightsquigarrow \pi_{\psi_v}^{GL_{2n+1}}$

$$\Pi_{\psi_v} \longrightarrow \mathrm{Rep}(\mathcal{S}_{\psi_v}) \qquad \mathcal{S}_{\psi_v} = \pi_0\left(Z_{\hat{G}}(\psi_v) / Z(\hat{G}) \right)$$

$$\pi_v \longmapsto \varepsilon_{\pi_v}$$

$$\mathcal{S}_\psi \longrightarrow \prod_v \mathcal{S}_{\psi_v} \qquad \varepsilon_\pi = \bigotimes_v \varepsilon_{\pi_v}$$

3. $m_\psi(\pi) = \dfrac{1}{|\mathcal{S}_\psi|} \displaystyle\sum_{s \in \mathcal{S}_\psi} \varepsilon_\pi(s) \; \varepsilon_\psi(s)$

defined by root numbers

$$\hat{G} = GSp(2n) \quad \text{e.g. } Sp(2) = SL(2),\ GSp(2) = GL(2)$$

$$1 \to G \to \tilde{G} \to \mathbb{G}_m \to 1$$

$$z \longmapsto z^2$$

$$\begin{tikzcd}
 & & & \mathbb{C}^\times & \\
1 \arrow[r] & \mathbb{C}^\times \arrow[r] \arrow[urr] & \tilde{G}^\vee \arrow[r] \arrow[ur] & \hat{G} \arrow[r] & 1 \\
 & \mathrm{Spin}(2n+1, \mathbb{C}) \arrow[ur] & GSpin(2n+1, \mathbb{C}) \arrow[u, phantom, "\parallel"] & &
\end{tikzcd}$$

Clifford algebra

Conj (Arthur): Fix $\tilde\eta : Z_{\tilde G}(\mathbb{A}_k)/Z_{\tilde G}(k) \longrightarrow \mathbb{C}^\times$

$$L^2_{disc}(\tilde G(k)\backslash \tilde G(\mathbb{A}_k), \tilde\eta) = \bigoplus_{\tilde\psi\in\Psi_2(\tilde G,\tilde\eta)} \bigoplus_{\tilde\pi\in\Pi_{\tilde\psi}} m_{\tilde\psi}(\tilde\pi)\,\tilde\pi$$

Spinor norm

$\mathcal{L}_\psi$

$$\begin{tikzcd}
\tilde\psi : L_k\times SL_2(\mathbb{C}) \arrow[r] \arrow[dr, "\psi"'] & \tilde G^\vee \arrow[r] \arrow[d] & \mathbb{C}^\times \\
& G^\vee &
\end{tikzcd}$$

$\eta_{\psi_1}\cdot\eta_{\psi_2} = 1$

$\psi = \psi_1 \boxplus \psi_2$

$\mathcal{S}_\psi \simeq \mathbb{Z}/2\mathbb{Z} \ni s$

$$Y = \mathrm{Hom}(\Gamma_k, \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Hom}(k^\times\backslash \mathbb{A}_k^\times, \mathbb{Z}/2\mathbb{Z})$$

$$\Psi_2(\tilde G,\tilde\eta) \xrightarrow{\simeq} \{(\psi,\omega) \mid \psi\in\Psi_2(G),\ \omega\in Y/\alpha(\mathcal{S}_\psi)\}$$

$$1 \longrightarrow \underset{}{\overset{1\,=}{\mathcal{S}_{\tilde\psi}}} \longrightarrow \mathcal{S}_\psi \xrightarrow{\alpha} Y$$

$1\neq s \longmapsto \eta_{\psi_1}\neq 1$

Rmk: · $\alpha(\mathcal{S}_\psi) = \{\omega\in Y \mid \tilde\psi\otimes\omega = \tilde\psi\}$

· α only depends on $\mathcal{L}_\psi$

$$\Pi_{\tilde\psi} = \bigotimes_v{}' \,\Pi_{\tilde\psi_v} \qquad \Pi_{\tilde\psi\otimes\omega} = \Pi_{\tilde\psi}\otimes\omega$$

$$\Pi_{\tilde{\psi}_v}\big|_{G(k_v)} = \Pi_{\psi_v}$$

$$\tilde{\Pi}_{\psi,\tilde{\zeta}} := \left\{ \tilde{\pi} \in \mathrm{Irr}(\tilde{G}(k_v)) \;\middle|\; \begin{array}{l} \tilde{\pi}|_{G(k_v)} \subseteq \Pi_{\psi} \\ \chi_{\tilde{\pi}} = \tilde{\zeta} \end{array} \right\}$$

$$\begin{tikzcd}
\Pi_{\tilde{\psi}} \subseteq \tilde{\Pi}_{\psi,\tilde{\zeta}} \arrow[d] \arrow[rr, dashed, "\text{well-defined}"] & & \mathrm{Rep}(\mathcal{S}_{\tilde{\psi}}) \\
\Pi_{\psi} \arrow[r] & \mathrm{Rep}(\mathcal{S}_{\psi}) \arrow[ur] &
\end{tikzcd}$$

$$m_{\tilde{\psi}}(\tilde{\pi}) = \frac{1}{|\mathcal{S}_{\tilde{\psi}}|} \sum_{s \in \mathcal{S}_{\tilde{\psi}}} \varepsilon_{\tilde{\pi}}(s)\, \underbrace{\varepsilon_{\tilde{\psi}}(s)}_{= \varepsilon_{\psi}|_{\mathcal{S}_{\tilde{\psi}}}}$$

Conj': For $\psi \in \Psi_2(G)$, $\exists\ \Pi_{\tilde{\psi}}$ unique up to twists by Y s.t

$$L^2_{disc}\left(\widetilde{G}(\mathbb{A}_k) / \widetilde{G}(k), \hat{\eta} \right) = \bigoplus_{\psi \in \Psi_2(G,\eta)} \bigoplus_{\omega \in Y/\alpha(\mathcal{L}_\psi)} \bigoplus_{\tilde{\pi} \in \Pi_{\tilde{\psi}} \otimes \omega} m_{\tilde{\psi}}(\tilde{\pi})\, \tilde{\pi}$$

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2. Results

F p-adic, $G = Sp(2n)/_F$

$\psi : WD_F \times SL_2(\mathbb{C}) \longrightarrow G^\vee$

2.1 $\psi = \phi$ tempered, i.e., trivial on $SL_2(\mathbb{C})$

Thm (X. 2016): $\exists\ \Pi_{\tilde{\phi}} \subseteq \widetilde{\Pi}_{\phi, \hat{\eta}}$ unique up to twists by $X := \mathrm{Hom}(F^\times, \mathbb{Z}/2\mathbb{Z})$ s.t

① $\widetilde{\Pi}_{\phi,\hat{\eta}} = \bigsqcup_{\omega \in X/\alpha(\mathcal{L}_\phi)} \Pi_{\tilde{\phi}} \otimes \omega$

$GSp(2n) \times GSO(2m_?)$

$GSp(2n) \quad \left(G(Sp(2n_1) \times SO(2n_2))\right)$

② $\eta_{\hat{\phi}} := \sum_{\hat{\pi} \in \Pi_{\hat{\phi}}} \hat{\pi}$ stable $\uparrow \quad \uparrow$

$Sp(2n) \quad Sp(2n_1) \times SO(2n_2)$

Rmk: · $\Pi_{\hat{\phi}}$ satisfies the character relation.

· we have shown if

$$S = \sum_{\hat{\pi} \in \tilde{\Pi}_{\phi, \zeta}} c_{\hat{\pi}} \tilde{\pi} \quad \text{stable}$$

then $S = \sum_{\omega} c_\omega \, \eta_{\hat{\phi}} \otimes \omega$

2.2 ψ nontempered

Thm (X.1): $\exists \ \Pi_{\hat{\psi}} \subseteq \tilde{\Pi}_{\psi, \zeta}$ s.t

① $\Pi_{\hat{\psi}} \big|_{G(F)} = \Pi_\psi$

② $\eta_{\hat{\psi}} = \sum_{\hat{\pi} \in \Pi_{\hat{\psi}}} \varepsilon_{\hat{\pi}}(s_{\hat{\psi}}) \tilde{\pi}$ stable

$\hat{\psi}\left(1 \times \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}\right)$

Rmk: · $\Pi_{\hat\psi}$ is constructed explicitly

· $\Pi_{\hat\psi}$ satisfies character relation

· $\Pi_{\hat\psi} \otimes \omega$, $\omega \in X / \alpha(\ell_\psi)$ are not disjoint

Thm (X.): If ψ satisfies certain combinatorial condition $(*)$, then $\Pi_{\hat\psi}$ in previous thm is unique up to twist by X. Moreover, if $S = \sum_{\hat\pi \in \Pi_{\psi,\hat\jmath}} c_{\hat\pi} \hat\pi$ stable, then

$$S = \sum_{\omega} c_\omega \, \eta_{\hat\psi} \otimes \omega$$

Rmk: This also implies any stable virtual representation from Π_ψ is a constant multiple of η_ψ.

3. Construction

3.1 F p-adic G connected reductive / F

$$\Psi(G) \hookrightarrow \Phi(G)$$

$$\psi \longmapsto \phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & \\ & |w|^{-\frac{1}{2}} \end{pmatrix}\right)$$

- $\Pi_\psi \supseteq \Pi_{\phi_\psi}$
- elements of Π_ψ have the same infinitesimal character

Assume $\mathrm{Irr}(G) \xrightarrow{\text{LLC}} \Phi(G)$

$$\pi \longmapsto \phi_\pi$$

For $\phi : W_F \times SL_2(\mathbb{C}) \longrightarrow G^\vee$

infinitesimal character $\lambda_\phi : W_F \longrightarrow G^\vee$

$$w \longmapsto \phi\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & \\ & |w|^{-\frac{1}{2}} \end{pmatrix}\right)$$

Fix λ, $\mathrm{Irr}(G)_\lambda := \{ \pi \in \mathrm{Irr}(G) \mid \lambda_{\phi_\pi} = \lambda \}$

$$\Phi(G)_\lambda \xrightarrow{\simeq} H_\lambda\text{-orbit on } V_\lambda$$

$$\phi \longmapsto C_\phi$$

$H_\lambda = Z_{G^\vee}(\lambda)$

$V_\lambda =$ q-eigenspace of $\mathrm{Ad}(\lambda(Fr))$ on $\mathfrak{g}^{\vee\,\lambda(I_F)}$

Conj: $\Pi_\psi \subseteq \{ \pi \in \mathrm{Irr}(G)_{\lambda_{\phi_\psi}} \mid \overline{C}_{\phi_\pi} \supseteq C_{\phi_\psi} \}$

due to ABV, V, CFMMX — not known

Rmk: $G = Sp(2n)$, LLC is known (Arthur)

$\Pi_\psi \subseteq \mathrm{Irr}(G)_{\lambda_{\phi_\psi}}$ (Moeglin)

Next, $\tilde{G} = GSp(2n)$

$\tilde{\lambda} : W_F \longrightarrow \tilde{G}^\vee$

$$\lambda \searrow \qquad \downarrow$$
$$G^\vee$$

$$\begin{array}{ccc} V_{\tilde\lambda} & \xrightarrow{\;\cong\;} & V_\lambda \\ H_{\tilde\lambda}/Z(\tilde G^\vee) & \hookrightarrow & H_\lambda \end{array} \quad \Rightarrow \quad \Phi(\tilde G)_{\tilde\lambda} \xrightarrow{(\Delta)} \Phi(G)_\lambda$$

identity component

surjection

Conj: $\Pi_{\tilde\varphi} = \left\{ \tilde\pi \in \mathrm{Irr}(\tilde G)_{\tilde\lambda} \;\middle|\; \begin{array}{l} \tilde\pi|_G \subseteq \Pi_\varphi \\ \overline{C}_{\phi_{\tilde\pi}} \supseteq C_{\phi_{\tilde\varphi}} \end{array} \right\}$

can be dropped if (Δ) is bijective

"LLC" for $\tilde G$:

$$\tilde\varphi \longmapsto \Pi_{\tilde\varphi} \quad \text{up to twists}$$

Idea: make a choice s.t it satisfies

Conj (LLC$^+$): $\widetilde{G} \supseteq \widetilde{P} \supseteq \widetilde{M}$

$\widehat{\lambda}$: $W_F \xrightarrow{\widetilde{\lambda}_{\widetilde{M}}} \widehat{M}^\vee \longrightarrow \widetilde{G}^\vee$

$$\mathrm{Ind}_{\widetilde{P}}^{\widehat{G}} \mathrm{Irr}(\widehat{M})_{\widehat{\lambda}_{\widetilde{M}}} \subseteq \mathrm{Irr}(\widetilde{G})_\lambda$$

due to T. Heiner, known for G (Monsseoui)

Thm (X.): ① $\exists\, \widehat{\varphi} \longmapsto \Pi_{\widehat{\varphi}}$ s.t LLC$^+$ holds.

② If (∗) is bijective, then

$$\eta_{\widehat{\varphi}} = \sum_{\widetilde{\pi} \in \Pi_{\widehat{\varphi}}} \varepsilon_{\widetilde{\pi}}(s_{\widehat{\varphi}})\, \widetilde{\pi} \text{ stable.}$$

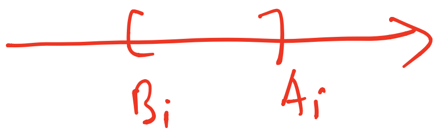
3.2 General case, i.e. (∗) is not necessarily injective

ψ : $W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow G^\vee \overset{\text{std}}{\hookrightarrow} GL_{2n+1}(\mathbb{C})$

Assume $\psi = \bigoplus_i \rho \otimes \mathrm{Sym}^{a_i-1} \otimes \mathrm{Sym}^{b_i-1}$

- $\rho \xrightarrow{\text{LLC}}$ self-dual irreducible unitary supercuspidal representation of $GL_{d_\rho}(F)$, orthogonal type

ψ



- $a_i + b_i$ even, $a_i > b_i$

- $(a_i, b_i) \longmapsto (A_i, B_i)$: $\begin{cases} A_i = \frac{a_i+b_i}{2} - 1 \\ B_i = \frac{a_i - b_i}{2} \end{cases}$

$[B_i, A_i]$ disjoint

Rmk: If $A_i = B_i$ for all i, then ψ π tempered.

Thm (Moeglin–Waldspurger): Fix (A, B), $A \neq B$

$$\eta_\psi = \bigoplus_{C \in]B,A]} (-1)^{A-C} \, St(\rho, B+C+1) \, |\det(\cdot)|^{\frac{B-C}{2}}$$

$$\rtimes \, Jac_{B+2 \cdots C} \, (-1)^{?} \, \eta_{\psi_1}$$

$$\oplus \ (-1)^{\left[\frac{A-B+1}{2}\right]} (-1)^{?} \, \eta_{\psi_2}$$

ψ_1 : (A, B) replaced by $(A, B+2)$

ψ_2 : $\cdots \quad \cdots$ by $(A, B+1)$, (B, B)

Idea: construct $\Pi_{\tilde{\psi}}$ in a family by induction on $\sum_i A_i - B_i$.

Def:

$$\eta_{\tilde{\psi}} := \oplus \cdots \eta_{\tilde{\psi}_1} \oplus \underbrace{\boxed{\overbrace{\eta_\rho^{c}}^{\text{closure relation}} \ \Big| \ \overbrace{\omega_\rho^{\delta_B(c)} \, \chi_{\tilde{\rho}}}^{\text{infinitesimal}}}}$$

$$\oplus \cdots \eta_{\tilde{\psi}_2}$$

$\mathbb{C}$ $\exp(2\pi i)$

$\downarrow$

$\mathbb{G}_m$

- η_ρ is central character of ρ
- $\omega_\rho = 1 \cdot 1^{d_\rho}$, $\quad \delta_B(c) = \frac{B+1}{2} + \left((B+2) + \cdots + c \right)$

- $\chi_{\tilde{\rho}}$ is central character of $\Pi_{\tilde{\rho}}$.

Thm (X.): $\eta_{\tilde{\psi}} = \sum_{\pi \in \Pi_\psi / \tilde{G}} \varepsilon_\pi(s_\psi)\, \tilde{\pi}$

It follows the pf of Moeglin in case of G.

At last, the combinatorial condition (*) required for the uniqueness part, it is

$$A_i \geq A_{i-1}, \quad B_i \geq B_{i-1}$$

This suffices to lift $\Pi_{\tilde{\psi}}$ to global ones!

global

ψ simple

Π_{ψ_v}