TOWARD THE ENDOSCOPIC CLASSIFICATION OF UNIPOTENT REPRESENTATIONS OF p-ADIC G_2

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Abstract. In this paper, we review the Langlands correspondence for unipotent representations of the exceptional group of type G_2 over a p-adic field F and present it in an explicit form. Then we compute all ABV-packets, as introduced in [CFM+21] following ideas from [Vog93], for these representations and prove that these packets satisfy properties derived from the expectation that they are generalized A-packets. We attach distributions to ABV-packets for G_2 and its endoscopic groups and study a geometric endoscopic transfer of the distributions. This paper builds on [CFZ].

Contents

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0. Introduction

0.1. **Background and motivation.** Local Arthur packets are finite sets $\Pi_\psi(G(F))$ of irreducible admissible representations of a local group $G(F)$ characterized by endoscopic character identities, defined for symplectic group and quasi-split orthogonal groups G and their inner twists over a local field F in [Art13]. Arthur packets generalize the notion of L-packets and they remedy certain drawbacks of L-packets associated with stable distributions. These notions have been extended to some other groups, including unitary groups in [Mok15], but not yet to exceptional groups, though candidates for some local Arthur packets for $G_2(F)$ were proposed in [GGJ02], [GG05] and [GS04]. In this paper we use the microlocal geometry of the moduli space of unramified Langlands parameters to calculate ABV-packets, as defined in [CFM+21], for all unipotent representations of $G_2(F)$ and also to show that these packets are compatible with endoscopy in a very precise sense.

Vogan's work [Vog93] shows that there are advantages to studying the representation theory of $G(F)$ together with that of its pure inner forms $G^\delta(F)$, simultaneously, in which case one passes from L-packets $\Pi_\phi(G(F))$ to "pure" L-packets $\Pi^{\mathrm{pure}}_\phi(G)$ and from Arthur packets $\Pi_\psi(G(F))$ to "pure" Arthur packets $\Pi^{\mathrm{pure}}_\psi(G)$. When Arthur's work is viewed from this perspective, $\Pi_\psi(G(F)) \to \mathsf{Rep}(S_\psi)$ extends to a map $\Pi^{\mathrm{pure}}_\psi(G) \to \mathsf{Rep}(A_\psi)$ where A_ψ is the component group of $Z_{\widehat{G}}(\psi)$, as explained [CFM+21]. Although there are no pure inner forms of $G_2(F)$, its endoscopic groups do admit pure inner forms.

The geometric construction of Arthur packets for any p-adic group is proposed in [CFM+21] using vanishing cycles of perverse sheaves, building on ideas introduced by [Vog93] and also by an adaptation of ideas from [ABV92]. More precisely, for a given algebraic reductive group G over a p-adic field F and a local Langlands parameter ϕ of $G(F)$, Cunningham *et.al.* constructed a group A^{ABV}_ϕ, a packet $\Pi^{\mathrm{ABV}}_\phi(G)$ of irreducible representations of $G(F)$ and its pure inner forms, called the ABV-packet of ϕ, and a natural map $\Pi^{\mathrm{ABV}}_\phi(G) \to \mathsf{Rep}(A^{\mathrm{ABV}}_\phi)$. If ϕ happens to be of Arthur type ψ, then $A^{\mathrm{ABV}}_\phi = A_\psi$.

The main conjecture of [CFM+21] says that when ϕ is of Arthur type ψ then $\Pi^{\mathrm{ABV}}_\phi(G) = \Pi^{\mathrm{pure}}_\psi(G)$ and the map $\Pi^{\mathrm{ABV}}_\phi(G) \to \mathsf{Rep}(A^{\mathrm{ABV}}_\phi)$ should be the same as $\Pi_\psi(G) \to \mathsf{Rep}(A_\psi)$ when G is a classical group. The construction in [CFM+21] not only gives a concrete way to compute Arthur packets for classical groups, as illustrated by the numerous examples in [CFM+21], but also provides a method to generalize Arthur packets to exceptional groups and non-Arthur type local Langlands parameters which, at this moment, seems unreachable using the trace formula method.

0.2. **Fundamental properties of ABV-packets and packet coefficients.** To give some details of our results, we recall that, in [Lus95] Lusztig proved the local Langlands correspondence for unipotent representations and showed that the unipotent representations are classified by unramified parameters. In this paper, for each unramified local Langlands parameter ϕ of $G_2(F)$, we compute the ABV-packets defined in [CFM+21] and thus we get a decomposition:

$$\Pi(G_2(F))_{\text{unip}} = \bigcup_{\phi\in\Phi(G_2(F))_{\text{unr}}} \Pi^{\text{ABV}}_{\phi}(G_2(F)),$$

where $\Pi(G_2(F))_{\text{unip}}$ is the set of isomorphism classes of irreducible unipotent representations of $G_2(F)$. For every ABV-packet $\Pi^{\text{ABV}}_{\phi}(G_2(F))$, the general, canonical, geometric constructions in [CFM+21] also give a group A^{ABV}_{ϕ} and a function

$$\begin{array}{rcl} \Pi^{\text{ABV}}_{\phi}(G_2(F)) & \to & \mathsf{Rep}(A^{\text{ABV}}_{\phi}) \\ \pi & \mapsto & \langle\ ,\pi\rangle \end{array}$$

and thus a function $\langle\ ,\ \rangle : A^{\text{ABV}}_{\phi} \times \Pi^{\text{ABV}}_{\phi}(G_2(F)) \to \bar{\mathbb{Q}}$; we refer to this as the *ABV-packet coefficients* for ϕ.

In this paper we argue that ABV-packet coefficients provide a canonical way to extend the notion of Arthur packets to all unipotent representations of the exceptional p-adic group G_2, including those that are not of Arthur type. For instance, in Theorem 2.2 we show that the ABV-packet coefficients extend the local Langlands correspondence for unipotent representations of $G_2(F)$: for all unramified Langlands parameters $\phi : W'_F \to {}^LG_2$, the following diagram commutes,

$$\begin{array}{ccc} \Pi^{\text{ABV}}_{\phi}(G_2(F)) & \longrightarrow & \widehat{A^{\text{ABV}}_{\phi}} \\ \uparrow & & \uparrow \\ \Pi_{\phi}(G_2(F)) & \xrightarrow[\text{LLC}]{} & \widehat{A_{\phi}}, \end{array}$$

where $\widehat{A^{\text{ABV}}_{\phi}}$ (resp. $\widehat{A_{\phi}}$) denotes the set of characters of irreducible representations of A^{ABV}_{ϕ} (resp. A_{ϕ}). Here the local Langlands correspondence is normalized such that if $\pi \in \Pi_{\phi}(G_2(F))$ is generic or spherical then corresponding representation of A_{ϕ} is trivial. We give other fundamental properties of ABV-packet coefficients in Theorem 2.2.

0.3. **Distributions attached to ABV-packets.** We study distributions defined by ABV-packet coefficients. To do this we introduce an algebraic group $\mathcal{S}^{\text{ABV}}_{\phi}$ contained in $Z_{\widehat{G}}(\phi)$ such that $A^{\text{ABV}}_{\phi} = \pi_0(\mathcal{S}^{\text{ABV}}_{\phi})$ and $\mathcal{S}^{\text{ABV}}_{\phi} = Z_{\widehat{G_2}}(\psi)$ if ϕ is of Arthur type ψ. For $s \in \mathcal{S}^{\text{ABV}}_{\phi}$, we set

$$\Theta_{\phi,s} := \sum_{\pi\in\Pi^{\text{ABV}}_{\phi_\psi}(G_2(F))} (-1)^{\dim(\phi)-\dim(\pi)} \langle s,\pi\rangle\ \Theta_{\pi}, \tag{1}$$

as in Definition 4, where we identify s with its image of s under $\mathcal{S}^{\text{ABV}}_{\phi} \to A^{\text{ABV}}_{\phi}$ and where Θ_{π} is the Harish-Chandra distribution character determined by the admissible representation π. The terms $\dim(\phi)$ and $\dim(\pi)$ are defined in Section 2.2. We show that if ϕ is of Arthur type ψ then $(-1)^{\dim(\phi)-\dim(\pi)}\langle s,\pi\rangle = \langle s_\psi\, s,\pi\rangle$, where s_ψ is the image of $\psi(1,-1) \in Z_{\widehat{G}_2}(\psi)$; in this case, (1) takes the more familiar form

$$\Theta_{\psi,s} = \sum_{\pi\in\Pi^{\mathrm{ABV}}_{\phi_\psi}(G_2(F))} \langle s_\psi s, \pi\rangle \ \Theta_\pi.$$

In Theorem 2.14 we prove that if π is an irreducible unipotent representation of $G_2(F)$ then Θ_π can be expressed in terms of the distributions $\Theta_{\phi,s}$, letting ϕ range over unramified L-parameters with the same infinitesimal parameter as π and letting s range over representatives for the components of $\mathcal{S}^{\mathrm{ABV}}_\phi$. This is an important part of the endoscopic classification promised in the title of this paper. We also show that the span of the distributions $\Theta_{\phi,s}$, as s ranges over $\mathcal{S}^{\mathrm{ABV}}_\phi$, coincides with the span of the distributions Θ_π, as π ranges over Π^{ABV}_ϕ, if and only if $\Pi^{\mathrm{ABV}}_\phi(G_2(F)) \to \widehat{A^{\mathrm{ABV}}_\phi}$ is bijective. In this theorem we also show that if $\Theta_\phi := \Theta_{\phi,1}$ is stable when ϕ is elliptic (see Section 2.8), then Θ_ϕ is stable for all ϕ.

0.4. **Lifting stable distributions from endoscopic groups.** We show the ABV-packet coefficients $\langle \ , \ \rangle$ are compatible with the theory of endoscopy, in the following sense. Let (G,s,ξ) be an endoscopic triple for G_2. Unlike the group G_2, the endoscopic groups G may admit pure inner forms. Recall that a pure inner form for G is a cocycle $\delta\in Z^1(F,G)$ and that δ determines an inner form G^δ of G. We calculate the ABV-packets and ABV-packet coefficients for the endoscopic group G. We generalize the definition of the distributions above (1) from G_2 to the pure inner forms of endoscopic groups G for G_2. Let (G,s,ξ) be an endoscopic triple for G_2. In Definition 8 we introduce a "lifting" of stable distributions attached to ABV-packets for $G(F)$ to invariant distributions on $G_2(F)$ and prove Theorem 4.6: if $\phi : W'_F \to {}^LG$ is an unramified Langlands parameter that is ξ-conormal, in the sense of Definition 6, then

$$\mathrm{Lift}_{(G,s,\xi)}\,\Theta^G_\phi = \Theta_{\xi\circ\phi,s}, \tag{2}$$

for all $s\in\mathcal{S}^{\mathrm{ABV}}_\phi$, for $\mathcal{S}^{\mathrm{ABV}}_\phi$ given in Definition 1. We show that if ϕ is of Arthur type then it is always ξ-conormal. Since $G(F)$ is also an endoscopic group for the inner form $G^\delta(F)$ we also define "lifting" from stable distribution attached to ABV-packets for $G(F)$ to invariant distributions on $G^\delta(F)$ and show that if $\phi : W'_F \to {}^LG$ is an unramified Langlands parameter that is relevant to $G^\delta(F)$, in the sense of [Bor79], then the lift of Θ^G_ϕ is $e(\delta)\Theta^{G^\delta}_\phi$, where $e(\delta)$ is the Kottwitz sign. To illustrate Theorem 4.6 we calculate all the lifts to $G_2(F)$ of the stable distributions attached to unramified Langlands parameters for the endoscopic group $\mathrm{PGL}_3(F)$. Conjecture 1 predicts that $\Theta_{\xi\circ\phi,s}$ is the Langlands-Shelstad transfer of the stable distribution Θ^G_ϕ.

All this comes together in Theorem 4.7, in which we show that ABV-packet coefficients are uniquely characterized by the properties given in this paper. This result also presents the endoscopic classification of unipotent representations of $G_2(F)$ promised in the title of this paper, stated here only for tempered representations to simplify the exposition. Let π be a tempered unipotent representation of $G_2(F)$ with Langlands parameter ϕ. Every $s\in\mathcal{S}_\psi$ determines an endoscopic triple (G,s,ξ) and a factorization $\phi=\xi\circ\phi^s$, for a Langlands parameter ϕ^s for G. In this case, Theorem 4.7 gives

$$\Theta_\pi = \sum_{(G,s,\xi)} (-1)^{\dim(\phi^s)-\dim(\pi)} \frac{\overline{\langle s,\pi\rangle}}{|Z_{A_\phi}(s)|}\ \mathrm{Lift}_{(G,s,\xi)}\,\Theta^G_{\phi^s}, \tag{3}$$

where the sum is taken over equivalence classes of endoscopic triples (G,s,ξ) with $s\in\mathcal{S}^{\mathrm{ABV}}_\phi$ and where we identify s with its image under $\mathcal{S}^{\mathrm{ABV}}_\phi \to A^{\mathrm{ABV}}_\phi$ in the calculation of $Z_{A_\phi}(s)$. This result is generalized from tempered unipotent representations to a wider class of unipotent representations in Theorem 4.7.

0.5. **Next steps.** Before we can declare that the ABV-packets of this paper are indeed (generalized) Arthur packets for $G_2(F)$, and change the title of this paper to *The* Endoscopic classification..., the following two issues must be addressed: (1) we must show that Θ^G_ϕ is a stable distribution for every unramified Langlands ϕ of $G(F)$ of Arthur type, where $G(F)$ is an endoscopic group for $G_2(F)$; and (2) we must show that, if (G, s, ξ) is an endoscopic triple for $G_2(F)$, then

$$\Theta^G_\phi(f^s) = \Theta_{\xi\circ\phi,s}(f)$$

when $f \in \mathcal{H}(G_2(F))$ is the Langlads-Shelstad transfer of $f^s \in \mathcal{H}(G(F))$; in other words, we must show that $\Theta_{\xi\circ\phi,s}$ is the endoscopic lift of Θ^G_ϕ.

In fact, it is a conjecture of Vogan that (1) is true for all Langlands parameters. This is elementary for unramified parameters except when $G = G_2$ and, as we show in Theorem 2.14, in that case it is sufficient to show that Θ_ϕ is stable for the four elliptic, tempered, unipotent L-packets in Table 2.8.1; see also Remark 2.16. We are working on a proof of this statement. We will also prove a generalization of (2) that asks only that ϕ is s-conormal Definition 9.

0.6. **Geometry and arithmetic.** ABV-packet coefficients are defined by a microlocal analysis of the moduli space of unramified Langlands parameters, following [CFM+21], building in [Vog93]. As in [CFM+21] and [CFZ], we say that the *infinitesimal parameter* of a Langlands parameter $\phi : W'_F \to {}^LG_2$ is $\lambda_\phi : W_F \to {}^LG_2$ defined by $\lambda_\phi(w) = \phi(w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2}))$. There is a natural moduli space for Langlands parameters with the same infinitesimal parameter as ϕ, introduced by Vogan and used extensively in [CFM+21] and [CFZ]. This moduli space is denoted by V_{λ_ϕ} in this paper and it naturally carries an action of the reductive group $H_{\lambda_\phi} := Z_{\widehat{G_2}}(\lambda_\phi)$; it is, in particular, a prehomogeneous vector space. In this paper we review the bijection between simple objects in the category $\mathsf{Per}_{H_{\lambda_\phi}}(V_{\lambda_\phi})$ equivariant perverse sheaves on V_{λ_ϕ} and irreducible admissible representations of $G_2(F)$ for which the Langlands parameter has infinitesimal parameter λ_ϕ.

We show that the Aubert involution of admissible representations appears as the Fourier transform of equivariant perverse sheaves on the other side of this bijection. We also show how to recognize when a unipotent representation is tempered, Arthur type, unitary, generic, spherical or supercuspidal purely in terms of the geometry of the corresponding simple equivariant perverse sheaf. To explain this, we say that ϕ is *open* (resp. *closed*) if the $Z_{\widehat{G_2}}(\lambda_\phi)$-orbit of ϕ is open (resp. closed) in V_{λ_ϕ}. If π is tempered then its Langlands parameter is open but the converse is not true; in this sense, the notion of open parameters generalizes the notion of parameters bounded upon restriction to W_F. Returning to the claim that ABV-packet coefficients provide an extension of the Langlands correspondence and Arthur packets, we also prove other fundamental properties of ABV-packet coefficients in Theorem 2.2.

0.7. **Notation.** We will use "polar notation" $\chi = \mu\nu^a$ for complex characters χ of $F^\times$, where $\nu : F^\times \to \mathbb{C}^\times$ is the unramified character defined by $\nu(\varpi) = q$ and where $\mu : F^\times \to \mathbb{C}^\times$ is unitary and $a \in \mathbb{R}$. When subscripts are necessary, this notation becomes $\chi_i = \mu_i\nu^{a_i}$. We will use the notation θ_n for a fixed, unitary character of $F^\times$ of order n.

Similarly, we use the "q-polar notation" $z = uq^a$ for complex numbers $z \in \mathbb{C}^\times$ where u is unitary and $a \in \mathbb{R}$.

By abuse of notation, we also denote a fixed primitive complex n-th root-of-unity by θ_n and write ϑ_n for the corresponding character of $\langle\theta_n\rangle := \{1, \theta_n, \dots, \theta_n^{n-1}\}$, so $\vartheta_n(\theta_n) = \theta_n$. When the order n is understood by context, we will write ϑ for this character.

We write ε for the sign character of S_3 and ϱ for the reflection representation of S_3.

For a split group G over F with a fixed Borel B and torus T over F, we set $I^G(\sigma) := \mathrm{Ind}_{B(F)}^{G(F)}(\sigma)$, where σ is a representation of $T(F)$, inflated to $B(F)$. Also, if P is a parabolic subgroup of

G with Levi subgroup M, we set $I_P^G(\sigma) := \mathrm{Ind}_{P(F)}^{G(F)}(\sigma)$. When this induced representation is a standard module, we write $J^G(\sigma)$ for its Langlands quotient and $J_P^G(\sigma)$.

For an algebraic group G over F, denote by 1_G the trivial representation of $G(F)$ and St_G the Steinberg representation of $G(F)$.

For representations of $G_2(F)$, we follow the notations from [Mui97] except that the Steinberg representation of $\mathrm{GL}_2(F)$ is denoted by $\mathrm{St}_{\mathrm{GL}_2}$ instead of $\delta(1)$ as in [Mui97]. There are 4 cuspidal unipotent representations $G_2[1]$, $G_2[-1]$, $G_2[\theta_3]$ and $G_2[\theta_3^2]$ of $G_2(\mathbb{F}_q)$ appearing in [Car93, p.460]. For such a cuspidal unipotent representation σ of $G_2(\mathbb{F}_q)$, we also uses the shorthand $I_0(\sigma)$ for $\mathrm{cInd}_{G_2(\mathcal{O}_F)}^{G_2(F)}(\sigma)$. These supercuspidal representations also appear in [CO17], where they are denoted by $v_6 = I_0(G_2[1])$, $v_7 = I_0(G_2[-1])$, $v_8 = I_0(G_2[\theta_3])$ and $v_9 = I_0(G_2[\theta_3^2])$. The last two of these also appear in [Sav99, Section 4], where they are denoted by $\pi'[\nu]$ and $\pi'[\nu^2]$.

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1. Geometry of the moduli space of Langlands parameters for G_2

1.1. **Roots for G_2 and $\widehat{G_2}$.** Let T be a fixed maximal split torus of G_2, the corresponding set of roots are denoted by $R := R(G_2, T)$. We write W for the Weyl group for this root system R. Let $\gamma_1, \gamma_2 : T \to F^\times$ be a choice simple roots of G_2 with γ_1 short and γ_2 long so that the positive roots R are

$$\gamma_1, \gamma_2, \gamma_1 + \gamma_2, 2\gamma_1 + \gamma_2, 3\gamma_1 + \gamma_2, 3\gamma_1 + 2\gamma_2.$$

The coroots are denoted $R^\vee := R(G_2, T)^\vee$, with $\gamma_1^\vee, \gamma_2^\vee : F^\times \to T$ the coroots of γ_1 and γ_2. We fix the isomorphism $T \to F^\times \times F^\times$ by

$$t \mapsto ((2\gamma_1 + \gamma_2)(t), (\gamma_1 + \gamma_2)(t)).$$

We use the notation $m : F^\times \times F^\times \to T$ for the inverse of this isomorphism. We have

$$\gamma_1^\vee(a) = m(a, a^{-1}), \qquad \text{and} \qquad \gamma_2^\vee(a) = m(1, a).$$

We denote by $\widehat{G_2}$ the dual group of G_2 over $\mathbb{C}$ and $\widehat{T}$ the dual torus and let $\hat{R} := R(\widehat{G_2}, \widehat{T})$ be the roots of $\widehat{G_2}$ with the usual identification of coroots of G_2 with roots of $\widehat{G_2}$. We again write W for the Weyl group for the root system $\hat{R}$. Denote by $\hat{\gamma}_1 \in \hat{R}$ (resp. $\hat{\gamma}_2 \in \hat{R}$) the image of $\gamma_1^\vee$ (resp. $\gamma_2^\vee$) under this identification. Then $\widehat{G}_2$ is a complex reductive group of type G_2, with simple roots $\hat{\gamma}_1, \hat{\gamma}_2$. where $\hat{\gamma}_1$ is a long root of $\widehat{G}_2$ and $\hat{\gamma}_2$ is a short root of $\widehat{G}_2$. As above, we fix an isomorphism $\widehat{T} \to \mathbb{C}^\times \times \mathbb{C}^\times$ by

$$t \mapsto ((\hat{\gamma}_1 + 2\hat{\gamma}_2)(t), (\hat{\gamma}_1 + \hat{\gamma}_2)(t))$$

and we write $\widehat{m} : \mathbb{C}^\times \times \mathbb{C}^\times \to \widehat{T}$ for the inverse of this isomorphism. Note that we have

$$\hat{\gamma}_1(\widehat{m}(x, y)) = x^{-1}y^2, \qquad \text{and} \qquad \hat{\gamma}_2(\widehat{m}(x, y)) = xy^{-1}. \tag{4}$$

We will denote by $\hat{\gamma}_1^\vee, \hat{\gamma}_2^\vee : \mathbb{C}^\times \to \widehat{T}$ the coroots of $\hat{\gamma}_1$ and $\hat{\gamma}_2$. We have again that

$$\hat{\gamma}_1^\vee(a) = \widehat{m}(1, a), \quad \hat{\gamma}_2^\vee(a) = \widehat{m}(a, a^{-1}).$$

Observe that under the identification $R(G_2, T) = R(\widehat{G_2}, \widehat{T})^\vee$, we have $\gamma_1 = \hat{\gamma}_1^\vee$, $\gamma_2 = \hat{\gamma}_2^\vee$, $(\hat{\gamma}_1 + \hat{\gamma}_2)^\vee = 3\hat{\gamma}_1^\vee + \hat{\gamma}_2^\vee$, $(\hat{\gamma}_1 + 2\hat{\gamma}_2)^\vee = 3\hat{\gamma}_1^\vee + 2\hat{\gamma}_2^\vee$, $(\hat{\gamma}_1 + 3\hat{\gamma}_2)^\vee = \hat{\gamma}_1^\vee + \hat{\gamma}_2^\vee$ and $(2\hat{\gamma}_1 + 3\hat{\gamma}_2)^\vee = 2\hat{\gamma}_1^\vee + \hat{\gamma}_2^\vee$.

1.2. **Moduli spaces of Langlands parameters.** Set $W_F' := W_F \times \mathrm{SL}_2(\mathbb{C})$. Given a Langlands parameter $\phi : W_F' \to {}^L G_2$ its *infinitesimal parameter* $\lambda_\phi : W_F \to {}^L G_2$ is defined by $\lambda_\phi(w) = \phi(w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2}))$.

For a given infinitesimal parameter $\lambda : W_F \to {}^L G_2$, there are only a finite number of $\widehat{G_2}$-conjugacy classes of Langlands parameters ϕ with $\lambda = \lambda_\phi$. As explained in [CFM+21] more generally, each unramified infinitesimal parameter $\lambda : W_F \to {}^L G_2$ determines a prehomogeneous vector space

$$V_\lambda = \{x \in \mathrm{Lie}\, \widehat{G_2} \mid \mathrm{Ad}(\lambda(w))x = qx, \forall w \in W_F\},$$

equipped with an action of the group

$$H_\lambda := \{g \in \widehat{G_2} \mid \lambda(w) g \lambda(w)^{-1} = g, \forall w \in W_F\}.$$

acting via the adjoint action of $\widehat{G_2}$ on $\mathrm{Lie}\, \widehat{G_2}$. Using this action we can assume that $\lambda(\mathrm{Fr}) \in {}^L T$ and define

$$\hat{R}_\lambda := \{\hat{\gamma} \in R(\widehat{G}, \widehat{T}) \mid \mathrm{Lie}\, U_{\hat{\gamma}} \subseteq V_\lambda\},$$

where $U_{\hat{\gamma}} \subset \widehat{G_2}$ is the root subgroup for $\hat{\gamma}$. Then V_λ is a moduli space for all Langlands parameters ϕ with $\lambda_\phi = \lambda$ and $X_\lambda := \widehat{G_2} \times_{H_\lambda} V_\lambda$ is a moduli space for all Langlands parameters ϕ for which λ_ϕ is $\widehat{G_2}$-conjugate to λ. Note that $\widehat{G_2}$ acts naturally on X_λ while H_λ acts on V_λ.

These notions lead to the following classification of unramified infinitesimal parameters λ for $G_2(F)$.

Proposition 1.1. *For every unramified $\lambda : W_F \to {}^L G_2$, the set $\hat{R}_\lambda$ is W-conjugate to one of the following subsets of $\hat{R}$, also pictured in Table 1.2.1:*

- ***0*** $\hat{R}_\lambda = \emptyset$;
- ***1*** $\hat{R}_\lambda = \{\hat{\gamma}_1 + 2\hat{\gamma}_2\}$;
- ***2*** $\hat{R}_\lambda = \{2\hat{\gamma}_1 + 3\hat{\gamma}_2\}$;
- ***3*** $\hat{R}_\lambda = \{\hat{\gamma}_1, 2\hat{\gamma}_1 + 3\hat{\gamma}_2\}$;
- ***4*** $\hat{R}_\lambda = \{\hat{\gamma}_1, \hat{\gamma}_1 + 2\hat{\gamma}_2\}$;
- ***5*** $\hat{R}_\lambda = \{\hat{\gamma}_2, \hat{\gamma}_1 + \hat{\gamma}_2\}$;
- ***6*** $\hat{R}_\lambda = \{\hat{\gamma}_1, \hat{\gamma}_1 + 3\hat{\gamma}_2\}$;
- ***7*** $\hat{R}_\lambda = \{\hat{\gamma}_1, \hat{\gamma}_2\}$;
- ***8*** $\hat{R}_\lambda = \{\hat{\gamma}_1, \hat{\gamma}_1 + \hat{\gamma}_2, \hat{\gamma}_1 + 2\hat{\gamma}_2, \hat{\gamma}_1 + 3\hat{\gamma}_2\}$.

Each case above determines a moduli space V_λ of unramified Langlands parameters with infinitesimal parameter λ.

Proof. The possibilities for λ are classified by the associated prehomogeneous vector spaces V_λ as they appear in $\mathrm{Lie}\, \widehat{G_2}$ up to $\widehat{G}_2$-conjugacy.

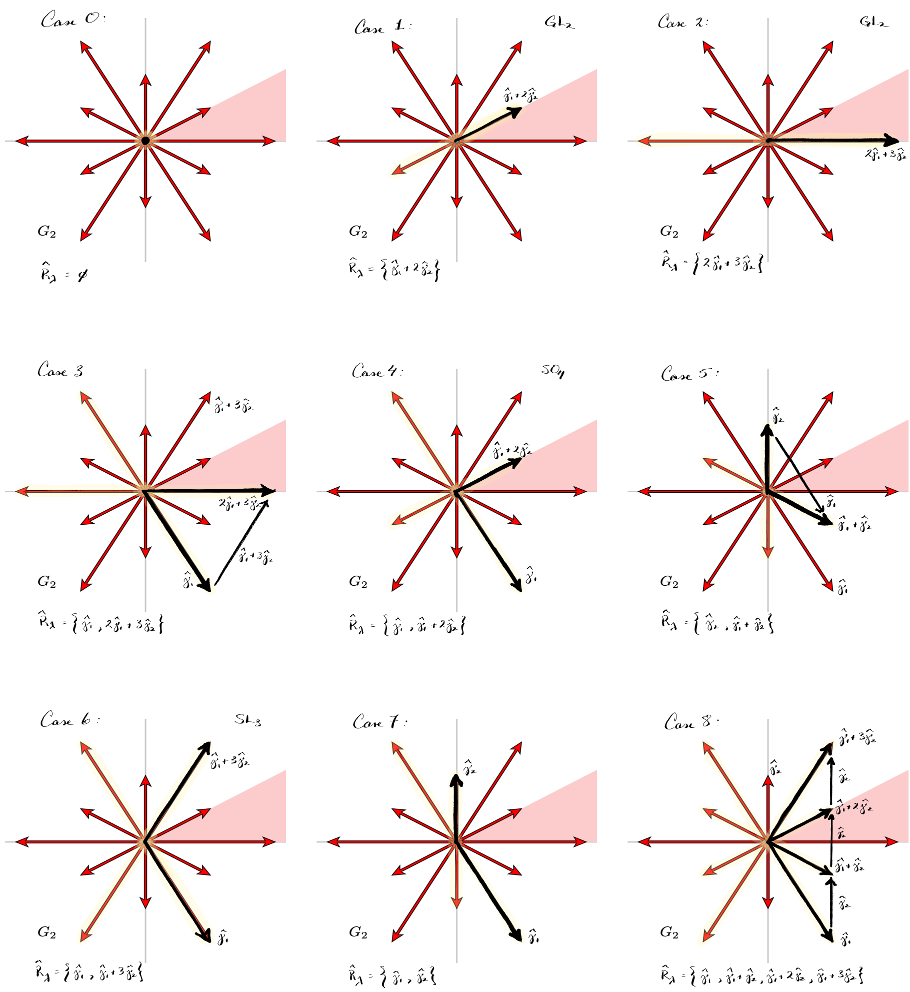
Case 0 describes all λ for which $V_\lambda = 0$. These are the λ for which $\hat{\gamma}(\lambda(\mathrm{Fr})) \neq q$ for every root $\hat{\gamma} \in \hat{R}$. In this case the group H_λ is $\widehat{G_2}$ or one of the subgroups $\mathrm{SL}_3(\mathbb{C})$, $\mathrm{SO}_4(\mathbb{C})$, $\mathrm{GL}_2^{\hat{\gamma}_1}(\mathbb{C})$, $\mathrm{GL}_2^{\hat{\gamma}_2}(\mathbb{C})$ or $\widehat{T}$, where $U_{\hat{\gamma}} \subset \mathrm{GL}_2^{\hat{\gamma}}$.

Cases 1 and 2 describe all λ for which $\dim V_\lambda = 1$; in these cases, $V_\lambda = \mathrm{Span}\{X_{\hat{\gamma}}\}$ for some root $\hat{\gamma}$ of $\widehat{G}_2$ and H_λ is either $\widehat{T}$ or $\mathrm{GL}_2^{\hat{\gamma}'}(\mathbb{C})$ for a root $\hat{\gamma}'$ perpendicular to $\hat{\gamma}$. Up to $\widehat{G}_2$-conjugacy, there are only two cases: either $\hat{\gamma}$ is short or long; in Case 1 we consider the short root $\hat{\gamma} = \hat{\gamma}_1 + 2\hat{\gamma}_2$, to which $\hat{\gamma}_1$ is perpendicular, and in Case 2 we consider the long root $\hat{\gamma} = 2\hat{\gamma}_1 + 3\hat{\gamma}_2$, to which $\hat{\gamma}_2$ is perpendicular.

Cases 3, 4, 5, 6 and 7 describe all λ for which $\dim V_\lambda = 2$; in these cases, $V_\lambda = \mathrm{Span}\{X_{\widehat{\gamma}}, X_{\widehat{\gamma}'}\}$ for roots $\widehat{\gamma}$ and $X_{\hat{\gamma}'}$ of $\widehat{G}_2$ and H_λ is either $\widehat{T}$ or $\mathrm{GL}_2^{\widehat{\gamma}_0}$ for a root $\hat{\gamma}_0$ such that $\widehat{\gamma}' = \widehat{\gamma} + \widehat{\gamma}_0$.

If $\dim V_\lambda > 2$ there is only one possibility for λ and this is Case 8, treated in [CFZ]. When $\dim V_\lambda > 2$ this forces $\dim V_\lambda = 4$ and that $V_\lambda = \mathrm{Span}\{X_{\widehat{\gamma}}, X_{\widehat{\gamma}+\widehat{\gamma}_0}, X_{\widehat{\gamma}+2\widehat{\gamma}_0}, X_{\widehat{\gamma}+3\widehat{\gamma}_0}\}$ for roots $\hat{\gamma}$ and $\widehat{\gamma}_0$. The group that acts on V_λ in this case is $H_\lambda = \mathrm{GL}_2^{\widehat{\gamma}_0}$. There are exactly six such V_λ, all $\widehat{G}_2$-conjugate to $\mathrm{Span}\{X_{\hat{\gamma}_1}, X_{\hat{\gamma}_1+\hat{\gamma}_2}, X_{\hat{\gamma}_1+2\hat{\gamma}_2}, X_{\hat{\gamma}_1+3\hat{\gamma}_2}\}$, in which case $H_\lambda = \mathrm{GL}_2^{\hat{\gamma}_2}$. ☐

Table 1.2.1. The classification of unramified infinitesimal parameters λ for $G_2(F)$ in terms of the subsets $\hat{R}_\lambda$ appearing in Proposition 1.1. Black root vectors with a halo appear in $\hat{R}_\lambda$ and so the corresponding root spaces form V_λ. Vectors in black without a halo determine root groups appearing in the reductive group H_λ acting on V_λ



1.3. **Prehomogeneous vector spaces.** This classification of infinitesimal parameters λ for G_2 by the associated prehomogeneous vector spaces $H_\lambda \times V_\lambda \to V_\lambda$ given in Proposition 1.1 may be simplified further by examining the categories $\mathsf{Per}_{H_\lambda}(V_\lambda)$ that arise as λ goes through the cases appearing there.

Proposition 1.2. *If $\lambda : W_F \to {}^L G_2$ is an unramified infinitesimal parameter for $G_2(F)$ then the category $\mathsf{Per}_{H_\lambda}(V_\lambda)$ is equivalent to $\mathsf{Per}_H(V)$ where $H \times V \to V$ is one of the following five prehomogeneous vector spaces.*

P0 *$V = 0$ and H is an algebraic group, not necessarily connected (trivial action);*

P1 *$V = \mathbb{A}^1$ with $H = \mathrm{GL}_1$-action given by $t \cdot x = tx$ (scalar multiplication);*
P2 *$V = \mathbb{A}^2$ with $H = \mathrm{GL}_2$-action $h \cdot x = det(h)^n hx$, for non-negative integer n (twisted matrix multiplication);*
P3 *$V = \mathbb{A}^2$ with $H = \mathrm{GL}_1^2$-action $(t_1, t_2) \cdot (x_1, x_2) = (t_1 x_1, t_1 t_2^n x_2)$, for positive integer n (toric variety);*
P4 *$V = \mathbb{A}^4$ with $H = \mathrm{GL}_2$-action $h \cdot x = (\det^{-1} \otimes \mathrm{Sym}^3)(h)(x) = \det(h)^{-1} \mathrm{Sym}^3(h)x$ (normalized Symmetric cube).*

Proof. We begin the proof with a general observation. Suppose H acts on V and let $u : H' \to H$ be an epimorphism of algebraic groups with connected kernel. Let H' act on V through u. Then $\mathsf{Per}_H(V)$ and $\mathsf{Per}_{H'}(V)$ are equivalent. To see this, recall that an H-equivariant perverse sheaf on V is a perverse sheaf $\mathcal{F}$ equipped with an isomorphism $\mu : m^*\mathcal{F} \to p^*\mathcal{F}$ satisfying a so-called cocycle condition. Consider the functor $\mathsf{Per}_H(V) \to \mathsf{Per}_{H'}(V)$ defined on objects by $(\mathcal{F}, \mu) \mapsto (\mathcal{F}, (u \times \mathrm{id})^*\mu)$ and on arrows by $\varphi \mapsto \varphi$. This functor is essentially surjective. Since $u \times \mathrm{id} : H' \times V \to H \times V$ is smooth with connected fibres of equal dimension $\dim \ker u$ the functor $(u \times \mathrm{id})^*$ is full and faithful by, for example, [BBD82, Proposition 4.2.5]. Thus, $\mathsf{Per}_H(V)$ and $\mathsf{Per}_{H'}(V)$ are equivalent. We refer to this equivalence as an instance of base-change, below, and in this case we will say that the prehomogeneous vector spaces $H \times V \to V$ and $H' \times V \to V$ are equivalent.

The cases below all refer to Proposition 1.1.

(Case 0) The prehomogeneous vector space in Proposition 1.1, Case 0 is $V = 0$ for a connected group acting on 0, which is an instance of the prehomogeneous vector space P0. By the base-change argument above, this is equivalent to P0 for trivial group H.

(Case 1) There are two group actions appearing in Proposition 1.1, Case 1, either $H_{\lambda_1} = \widehat{T}$ or $H_{\lambda_1} = \mathrm{GL}_2$. In the former case, observe that $\hat\gamma_1 + 2\hat\gamma_2 : \widehat{T} \to \mathrm{GL}_1$ is an epimorphism of algebraic groups with connected fibre so, by the paragraph above, $\mathsf{Per}_{H_{\lambda_1}}(V_{\lambda_1})$ is equivalent to $\mathsf{Per}_{\mathrm{GL}_1}(\mathbb{A}^1)$ for the prehomogeneous vector space P1. In the latter case, observe that $\det : \widehat{T} \to \mathrm{GL}_1$ is an epimorphism of algebraic groups with connected fibre so, by the base-change argument above $\mathsf{Per}_{H_{\lambda_1}}(V_{\lambda_1})$ is again equivalent to $\mathsf{Per}_{\mathrm{GL}_1}(\mathbb{A}^1)$ for the prehomogeneous vector space P1.

(Case 2) In Proposition 1.1, Case 2 there are two group actions to consider: either $H_{\lambda_2} = \widehat{T}$ or $H_{\lambda_2} = \mathrm{GL}_2$. Here, $V_{\lambda_2} = \mathbb{A}^1$. If $H_{\lambda_2} = \widehat{T}$ then the action is $h.x = \det(h)x$. Using the base-change argument above we have $\mathsf{Per}_{H_{\lambda_2}}(V_{\lambda_2}) = \mathsf{Per}_{\mathrm{GL}_1}(\mathbb{A}^1)$ since $2\hat\gamma_1 + 3\hat\gamma_2 : \widehat{T} \to \mathrm{GL}_1$ is an epimorphism of algebraic groups with connected fibre. If $H_{\lambda_2} = \mathrm{GL}_2$ then the action is $h \cdot x = \det(h)x$. Then, using the base-change argument above, $\mathsf{Per}_{H_{\lambda_2}}(V_{\lambda_2}) = \mathsf{Per}_{\mathrm{GL}_1}(\mathbb{A}^1)$ since $\det : \mathrm{GL}_2 \to \mathrm{GL}_1$ is an epimorphism of algebraic groups with connected fibre. Thus, $\mathsf{Per}_{H_{\lambda_2}}(V_{\lambda_2})$ is equivalent $\mathsf{Per}_H(V)$ for the prehomogeneous vector space P1.

(Case 3) The prehomogeneous vector space in Proposition 1.1, Case 3 is $V_{\lambda_3} = \mathbb{A}^2$ and $H_{\lambda_3} = \mathrm{GL}_2$; this is equivalent to the prehomogeneous vector space P2 for $n = 1$.

(Case 4) The prehomogeneous vector space in Proposition 1.1, Case 4 is $V_{\lambda_4} = \mathbb{A}^2$ and $H_{\lambda_4} = \widehat{T}$; this is equivalent to the prehomogeneous vector space P3 for $n = 2$.

(Case 5) The prehomogeneous vector space in Proposition 1.1, Case 5 is $V_{\lambda_5} = \mathbb{A}^2$ and $H_{\lambda_5} = \mathrm{GL}_2$; this is equivalent to the prehomogeneous vector space P2 for $n = 0$.

(Case 6) The prehomogeneous vector space in Proposition 1.1, Case 6 is $V_{\lambda_6} = \mathbb{A}^2$ and $H_{\lambda_6} = \widehat{T}$; this is equivalent to the prehomogeneous vector space P3 for $n = 3$.

(Case 7) The prehomogeneous vector space in Proposition 1.1, Case 7 is $V_{\lambda_7} = \mathbb{A}^2$ and $H_{\lambda_7} = \widehat{T}$; this is equivalent to the prehomogeneous vector space P3 for $n = 1$.

(Case 8) The prehomogeneous vector space in Proposition 1.1, Case 8 is $V_{\lambda_8} = \mathbb{A}^4$ and $H_{\lambda_8} = \mathrm{GL}_2$; as explained in [CFZ, Proposition 3.1], this is equivalent to the prehomogeneous vector space P4. Category $\mathsf{Per}_{\mathrm{GL}_2}(\det^{-1} \otimes \mathrm{Sym}^3)$ appeared in [CFZ]. □

1.4. **The geometry of the moduli space of Langlands parameters.** In this section we make a microlocal study of the five prehomogeneous vector spaces appearing in Proposition 1.2. In Section 2.4 we use this to determine the ABV-packets for $G_2(F)$.

Let $H \times V \to V$ be a prehomogeneous vector space. Let V^* be the dual vector space to V and let $H \times V^* \to V^*$ be the adjoint action. Then $H \times H$ acts on $T^*(V) = V \times V^*$ by $(h,k) \cdot (x,y) := (h \cdot x, k \cdot y)$ and $T^*(V)$ is also a prehomogeneous vector space. Now let $[\ ] : V \times V^* \to \mathrm{Lie}\, H$ be the momentum map and consider the conormal variety

$$\Lambda := \{(x,y) \in T^*(V),\ [x,y] = 0\}.$$

This Lagrangian subspace $\Lambda \subseteq T^*(V)$ carries the diagonal H-action $h \cdot (x,y) := (h \cdot x, h \cdot y)$. This is a bundle of vector spaces with group actions: for each $x \in V$, the fibre

$$\Lambda_x = \{y \in V^*,\ [x,y] = 0\}$$

caries a natural action of $Z_H(x)$. It is not always the case that Λ_x is a prehomogeneous vector space.

Proposition 1.3. *If V is one of the 5 prehomogeneous vector spaces appearing in Proposition 1.2 then Λ is a bundle of prehomogeneous vector spaces.*

When the conormal variety Λ to V is a bundle of prehomogeneous vector spaces, we write $\Lambda_x^{\mathrm{sreg}}$ for the open dense $Z_H(x)$-orbit in Λ_x and Λ_C^{reg} for the open dense H-orbit in Λ_C, for each H-orbit $C \subseteq V$.

In this section we study the conormal varieties to the prehomogeneous vector spaces appearing in Proposition 1.2. In each case we:

- find all H-orbits $C \subseteq V$ and all dual H-orbits $C^* \subseteq V^*$;
- calculate each equivariant fundamental group $A_C := \pi_0(Z_H(x))$, for $x \in C$;
- enumerate the simple objects in $\mathsf{Per}_H(V)$;
- find the Fourier transform of each simple equivariant perverse sheaf on V;
- show that the conormal variety Λ is a bundle of prehomogeneous vector spaces;
- calculate the equivariant fundamental group A_C^{ABV} of $\Lambda_C^{\mathrm{sreg}}$, for each H-orbit C in V;
- calculate $\mathsf{NEvs}_{C_j}\, \mathcal{IC}(\mathcal{L}_{C_i}) \in \mathsf{Loc}_H(\Lambda_{C_j}^{\mathrm{sreg}})$ for every simple $\mathcal{IC}(\mathcal{L}_{C_i})$ in $\mathsf{Per}_H(V)$, where NEvs is defined in [CFM+21, Section 7.10].

These geometric calculations, and their applications to representation theory, follow the ideas presented in the examples treated in [CFM+21, Part II] and in [CFZ].

(P0) $\mathsf{Per}_H(0)$ is the category of finite dimensional representations of $\pi_0(H)$. Then $\Lambda = 0$ and the functor Evs is the identity.

(P1) For the action of GL_1 on $\mathbb{A}^1$ given by scalar multiplication there are two GL_1-orbits: $C_- = \{0\}$ and its complement, C_1. Then $\mathsf{Per}_{\mathrm{GL}_1}(\mathbb{A}^1)$ has two simple objects, $\mathcal{IC}(\mathbb{1}_{C_0}) = \mathbb{1}_{C_0}^{!}[0]$, the extension-by-zero of $\mathbb{1}_{C_0}$ and $\mathcal{IC}(\mathbb{1}_{C_1}) = \mathbb{1}_{\mathbb{A}^1}[1]$. The conormal varieties are given as follows. First observe that $\Lambda_0 = T_0(\mathbb{A}^1) \cong \mathbb{A}^1$ and $\Lambda_0^{\mathrm{sreg}} \cong \{y \in \mathbb{A}^1 \mid y \neq 0\}$; note that this is a single orbit under the GL_1-action. If $x \in C_1$ then $\Lambda_x = \{(x,y) \mid xy = 0\} = \{(x,0)\}$ which itself is a GL_1-orbit. In this way we see that $T^*_{C_0}(\mathbb{A}^1)^{\mathrm{sreg}}$ and $T^*_{C_1}(\mathbb{A}^1)^{\mathrm{sreg}}$ are both isomorphic, as GL_1-spaces, to C_1 and consequently, equivariant local systems on these spaces are naturally identified with finite-dimensional vector spaces. The functor

$\mathsf{NEvs} : \mathsf{Per}_{\mathrm{GL}_1}(\mathbb{A}^1) \to \mathsf{Loc}_{\mathrm{GL}_1}(T^*_{\mathrm{GL}_1}(\mathbb{A}^1)_{\mathrm{sreg}})$ is given by the following table:

$\mathsf{Per}_{\mathrm{GL}_1}(\mathbb{A}^1)$	$\mathsf{Loc}_{\mathrm{GL}_1}(\Lambda^{\mathrm{sreg}}_{C_0})$	$\mathsf{Loc}_{\mathrm{GL}_1}(\Lambda^{\mathrm{sreg}}_{C_1})$	Fourier
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_0}}$	0	$\mathcal{IC}(\mathbb{1}_{C_1})$
$\mathcal{IC}(\mathbb{1}_{C_1})$	0	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_1}}$	$\mathcal{IC}(\mathbb{1}_{C_0})$

(P2) For the twisted matrix multiplication action of GL_2 on $\mathbb{A}^2$, again there are only two H-orbits, $C_0 = \{(0,0)\}$ and the complement C_1 and only two simple objects in $\mathsf{Per}_{\mathrm{GL}_2}(\mathbb{A}^2)$ are $\mathcal{IC}(\mathbb{1}_{C_0}) = \mathbb{1}^!_{C_0}[0]$ and $\mathcal{IC}(\mathbb{1}_{C_1}) = \mathbb{1}_{\mathbb{A}^2}[2]$. Again, $T^*_{C_0}(\mathbb{A}^2)^{\mathrm{sreg}}$ and $T^*_{C_1}(\mathbb{A}^2)^{\mathrm{sreg}}$ are both isomorphic, as GL_2-spaces, to $\bar{C}_1$, which is a single GL_2-orbit and also has trivial equivariant fundamental group. The functor $\mathsf{NEvs} : \mathsf{Per}_{\mathrm{GL}_2}(\mathbb{A}^2) \to \mathsf{Loc}_{\mathrm{GL}_2}(T^*_{\mathrm{GL}_2}(\mathbb{A}^2)_{\mathrm{sreg}})$ is given by

$\mathsf{Per}_{\mathrm{GL}_2}(\mathbb{A}^2)$	$\mathsf{Loc}_{\mathrm{GL}_2}(\Lambda^{\mathrm{sreg}}_{C_0})$	$\mathsf{Loc}_{\mathrm{GL}_2}(\Lambda^{\mathrm{sreg}}_{C_1})$	Fourier
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_0}}$	0	$\mathcal{IC}(\mathbb{1}_{C_1})$
$\mathcal{IC}(\mathbb{1}_{C_1})$	0	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_1}}$	$\mathcal{IC}(\mathbb{1}_{C_0})$

(P3) Consider the action of $H = \mathrm{GL}_1^2$ on $V = \mathbb{A}^2$ by $(t_1,t_2).(x_1,x_2) = (t_1x_1, t_1t_2^n x_2)$ for positive integer n. There are four H-orbits in this case: $C_0 = \{(0,0)\}$, $C_1 = \{(x_1,0) \mid x_1 \neq 0\}$, $C_2 = \{(0,x_2) \mid x_2 \neq 0\}$ and the open orbit $C_3 = \{(x_1,x_2) \mid x_1x_2 \neq 0\}$. The equivariant fundamental groups of these orbits are trivial with the exception of C_3, for which A_{C_3} is the group of n-th roots-of-unity. There are $4 + (n-1)$ simple objects in this category: $\mathcal{IC}(\mathbb{1}_{C_0}) = \mathbb{1}^!_{C_0}$, $\mathcal{IC}(\mathbb{1}_{C_1}) = \mathbb{1}^!_{\bar{C}_1}[1]$, $\mathcal{IC}(\mathbb{1}_{C_2}) = \mathbb{1}^!_{\bar{C}_2}[1]$, $\mathcal{IC}(\mathbb{1}_{C_3}) = \mathbb{1}_{\mathbb{A}^2}[2]$ and $\mathcal{IC}(\vartheta_{C_3})$, where ϑ is a non-trivial character of the group of nth roots-of-unity. For $x \in C_0$ or C_3 we have $\Lambda_x \cong \bar{C}_3$ which has a unique open GL_1^2-orbit. For $x \in C_1$, $\Lambda_x = \{((x_1,0),(y_1,y_2)) \in T^*(V) \mid x_1y_1 = 0\} = \{((x_1,0),(0,y_2)) \mid y_2\} \cong \bar{C}_2$ which also has a unique GL_1^2-orbit; likewise for $x \in C_2$, $\Lambda_x \cong \bar{C}_1$. It follows that Λ_C has a unique open H-orbit, $\Lambda^{\mathrm{sreg}}_C$, for every H-orbit $C \subset V$. A simple calculation shows that the equivariant fundamental group of $\Lambda^{\mathrm{sreg}}_C$ is the group $\langle\theta_n\rangle$ of nth roots-of-unity, for every H-orbit $C \subset V$. The functor NEvs is given by the following table, where the last row is removed if $n = 1$ and duplicated if $n > 2$ for every non-trivial character ϑ.

$\mathsf{Per}_{\mathrm{GL}_1^2}(\mathbb{A}^2)$	$\mathsf{Loc}_{\mathrm{GL}_1^2}(\Lambda^{\mathrm{sreg}}_{C_0})$	$\mathsf{Loc}_{\mathrm{GL}_1^2}(\Lambda^{\mathrm{sreg}}_{C_1})$	$\mathsf{Loc}_{\mathrm{GL}_1^2}(\Lambda^{\mathrm{sreg}}_{C_2})$	$\mathsf{Loc}_{\mathrm{GL}_1^2}(\Lambda^{\mathrm{sreg}}_{C_3})$	Fourier
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_0}}$	0	0	0	$\mathcal{IC}(\mathbb{1}_{C_3})$
$\mathcal{IC}(\mathbb{1}_{C_1})$	0	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_1}}$	0	0	$\mathcal{IC}(\mathbb{1}_{C_2})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	0	0	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_2}}$	0	$\mathcal{IC}(\mathbb{1}_{C_1})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	0	0	0	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_3}}$	$\mathcal{IC}(\mathbb{1}_{C_0})$
$\mathcal{IC}(\vartheta_{C_3})$	$\vartheta_{\Lambda^{\mathrm{sreg}}_0}$	$\vartheta_{\Lambda^{\mathrm{sreg}}_1}$	$\vartheta_{\Lambda^{\mathrm{sreg}}_1}$	$\vartheta_{\Lambda^{\mathrm{sreg}}_2}$	$\mathcal{IC}(\vartheta_{C_3})$

(P4) The functor NEvs for $H = \mathrm{GL}_2$ acting on $V = \mathbb{A}^4$ by $h.x = \det^{-1}(h)\,\mathrm{Sym}^3(h)(x)$ (nomalized Symmetric cube) was calculated in [CFZ]. We recall the result in the table below, in which $\vartheta := \vartheta_2$ is the sign character of $\langle\theta_2\rangle$ and, as explained in Section 0.7, ε is the sign character of S_3 and ϱ is the character of the reflection representation of S_3.

$\mathsf{Per}_{\mathrm{GL}_2}(V)$	$\mathsf{Loc}_{\mathrm{GL}_2}(\Lambda^{\mathrm{sreg}}_{C_0})$	$\mathsf{Loc}_{\mathrm{GL}_2}(\Lambda^{\mathrm{sreg}}_{C_1})$	$\mathsf{Loc}_{\mathrm{GL}_2}(\Lambda^{\mathrm{sreg}}_{C_2})$	$\mathsf{Loc}_{\mathrm{GL}_2}(\Lambda^{\mathrm{sreg}}_{C_3})$	Fourier
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{\Lambda^{\mathrm{reg}}_0}$	0	0	0	$\mathcal{IC}(\mathbb{1}_{C_3})$
$\mathcal{IC}(\mathbb{1}_{C_1})$	$\varrho_{\Lambda^{\mathrm{reg}}_0}$	$\mathbb{1}_{\Lambda^{\mathrm{reg}}_1}$	0	0	$\mathcal{IC}(\varrho_{C_3})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	0	$\vartheta_{\Lambda^{\mathrm{reg}}_1}$	$\mathbb{1}_{\Lambda^{\mathrm{reg}}_2}$	0	$\mathcal{IC}(\mathbb{1}_{C_2})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	0	0	0	$\mathbb{1}_{\Lambda^{\mathrm{reg}}_3}$	$\mathcal{IC}(\mathbb{1}_{C_1})$
$\mathcal{IC}(\varrho_{C_3})$	0	0	$\vartheta_{\Lambda^{\mathrm{reg}}_2}$	$\varrho_{\Lambda^{\mathrm{reg}}_3}$	$\mathcal{IC}(\mathbb{1}_{C_0})$
$\mathcal{IC}(\varepsilon_{C_3})$	$\varepsilon_{\Lambda^{\mathrm{reg}}_0}$	$\mathbb{1}_{\Lambda^{\mathrm{reg}}_1}$	$\vartheta_{\Lambda^{\mathrm{reg}}_2}$	$\varepsilon_{\Lambda^{\mathrm{reg}}_3}$	$\mathcal{IC}(\varepsilon_{C_3})$

2. Unipotent representations for G_2

In this section we use the classification of infinitesimal parameters for $G_2(F)$ in Proposition 1.1, together with the results of Section 1.4, to enumerate all irreducible unipotent representations of $G_2(F)$ and to find their Langlands parameters. In this way we give an explicit form of the Langlands correspondence for unipotent representations of $G_2(F)$. At the same time, we will find the ABV-packet coefficients and start assembling the proof of Theorem 2.2.

2.1. The Langlands correspondence for unipotent representations of $G_2(F)$.

The Langlands correspondence for unipotent representations of $G_2(F)$ is a special case [Lus95], though the map $\Pi_\phi(G_2(F)) \to \widehat{A_\phi}$ given there is normalized differently. Here we are making our normalization of LLC such that it satisfies the desiderata of [GGP12, §9]. In particular, it satisfies [GP92, Conjecture 2.6], which specifies the Langlands parameter of generic representations for a particular infinitesimal parameter. Moreover, for a choice of hyperspecial group, our normalization also specifies the enhanced Langlands parameter of spherical representation for a particular infinitesimal parameter, i.e., if $\pi \in \Pi_\phi(G_2(F))$ is spherical, then ϕ is trivial on the $\mathrm{SL}_2(\mathbb{C})$ part and the corresponding representation of $\widehat{A}_\phi$ is trivial. This choice of normalization agrees with the requirement in the work of Arthur [Art13].

The Langlands correspondence for unipotent representations of $G_2(F)$ is given in Table 2.1.1, also in Table 2.5.1, making reference to the classification of unramified infinitesimal parameters in Proposition 1.1 and the enumeration of unramified Langlands parameters for $G_2(F)$ given in Section 2.4. Both tables use notation from [Mui97] for irreducible admissible representations of $G_2(F)$, modified as explained in Section 0.7. In Table 2.1.1, characters of A_ϕ are listed in the column with ϕ at the top and the L-packet $\Pi_\phi(G_2(F))$ is found by gathering together the representations π for which the character of A_ϕ is non-zero, this defining the bijection

$$\Pi_\phi(G_2(F)) \to \widehat{A_\phi},$$

for all unramified Langlands parameters for $G_2(F)$.

There are precisely three non-singleton L-packets of unipotent representations of $G_2(F)$: $\Pi_{\phi_{4d}}(G_2(F))$, $\Pi_{\phi_{6d}}(G_2(F))$ and $\Pi_{\phi_{8d}}(G_2(F))$, of which the first has order 2 and the other two have order 3. Note that the group A_ϕ is trivial except when ϕ is ϕ_{4d}, ϕ_{6d} or ϕ_{8d}, in which cases A_ϕ is $\langle\theta_2\rangle$, $\langle\theta_3\rangle$ and S_3, respectively. These three Langlands parameters are all of Arthur type and are distinguished by the fact that they have elliptic endoscopy, as explained in Section 4.

Recall that we have classified all unramified infinitesimal parameters for $G_2(F)$ in Section 1.2. Working through the 9 cases in Proposition 1.1, in Section 2.4 we find, in each case, all Langlands parameters ϕ with given infinitesimal parameter λ. Using results from [CFM+21], we compute the component group A_ϕ. We find that these groups are trivial in all cases except the unique tempered Langlands parameter with infinitesimal parameters given by Cases 4, 6 and 8. In all other cases, except these three, since A_ϕ is trivial, we use the Langlands classification to find the corresponding admissible representation and note that $\Pi_\phi(G_2(F)) \to \mathsf{Rep}(A_\phi)$ is trivial in these cases. This is done case-by-case in Section 2.4. It then remains to consider only the Langlands parameters with infinitesimal parameters given by Cases 4, 6 and 8, specifically the three tempered (*i.e.*, bounded upon restriction to W_F) parameters denoted by ϕ_{4d}, ϕ_{6d} and ϕ_{8d}. These three are of Arthur type and are lifted from Arthur parameters of elliptic endoscopic groups, as explained in Section 4. We found the local Langlands correspondence for parameters with infinitesimal parameter λ_8 in [CFZ, Theorem 2.5]. The local Langlands correspondence for parameters with infinitesimal parameter λ_4 and λ_6 is obtained by similar arguments.

2.2. Vogan's version of the local Langlands correspondence.

Let λ be an unramifed infinitesimal parameter of $G_2(F)$ and let $\Pi_\lambda(G_2(F))$ be the set of (equivalence classes of) irreducible smooth representations of $G_2(F)$ with infinitesimal parameter λ. Associated with λ,

TABLE 2.1.1. The local Langlands correspondence for unipotent representations of $G_2(F)$. See Section 2.1 for how to read this table.

	λ_0	λ_1		λ_2		λ_3		λ_4				λ_5		λ_6				λ_7				λ_8			
	T	T	GL_2	T	GL_2	T	GL_2	T	GL_2	GL_2	SO_4	T	GL_2	T	GL_2	GL_2	PGL_3	T	GL_2	GL_2	G_2	T	GL_2	GL_2	PGL_3 SO_4
	ϕ_0	ϕ_{1a}	ϕ_{1b}	ϕ_{2a}	ϕ_{2b}	ϕ_{3a}	ϕ_{3b}	ϕ_{4a}	ϕ_{4b}	ϕ_{4c}	ϕ_{4d}	ϕ_{5a}	ϕ_{5b}	ϕ_{6a}	ϕ_{6b}	ϕ_{6c}	ϕ_{6d}	ϕ_{7a}	ϕ_{7b}	ϕ_{7c}	ϕ_{7d}	ϕ_{8a}	ϕ_{8b}	ϕ_{8c}	ϕ_{8d}
$I(\mu_1\nu^{a_1}\otimes\mu_2\nu^{a_2})$	1																								
$I_{\gamma_2}(a-1/2,\mu\circ\det)$		1	0																						
$I_{\gamma_2}(a-1/2,\mu\otimes\mathrm{St}_{\mathrm{GL}_2})$		0	1																						
$I_{\gamma_1}(a-1/2,\mu\circ\det)$				1	0																				
$I_{\gamma_1}(a-1/2,\mu\otimes\mathrm{St}_{\mathrm{GL}_2})$				0	1																				
$I_{\gamma_1}(1/6,\theta_3^n\circ\det)$						1	0																		
$I_{\gamma_1}(1/6,\theta_3^n\otimes\mathrm{St}_{\mathrm{GL}_2})$						0	1																		
$J_{\gamma_2}(1,I^{\mathrm{GL}_2}(1\otimes\theta_2))$								1	0	0	0														
$J_{\gamma_1}(1/2,\theta_2\otimes\mathrm{St}_{\mathrm{GL}_2})$								0	1	0	0														
$J_{\gamma_2}(1/2,\theta_2\otimes\mathrm{St}_{\mathrm{GL}_2})$								0	0	1	0														
$\pi(\theta_2)$								0	0	0	1														
$I_0(G_2[-1])$								0	0	0	ϑ_2														
$I_{\gamma_2}(3/2,1_{\mathrm{GL}_2})$												1	0												
$I_{\gamma_2}(3/2,\mathrm{St}_{\mathrm{GL}_2})$												0	1												
$J_{\gamma_2}(1,I^{\mathrm{GL}_2}(\theta_3\otimes\theta_3^{-1}))$														1	0	0	0								
$J_{\gamma_1}(1/2,\theta_3^2\otimes\mathrm{St}_{\mathrm{GL}_2})$														0	1	0	0								
$J_{\gamma_1}(1/2,\theta_3\otimes\mathrm{St}_{\mathrm{GL}_2})$														0	0	1	0								
$\pi(\theta_3)$														0	0	0	1								
$I_0(G_2[\theta_3])$														0	0	0	ϑ_3								
$I_0(G_2[\theta_3^2])$														0	0	0	ϑ_3^2								
1_{G_2}																		1	0	0	0				
$J_{\gamma_1}(3/2,\mathrm{St}_{\mathrm{GL}_2})$																		0	1	0	0				
$J_{\gamma_2}(5/2,\mathrm{St}_{\mathrm{GL}_2})$																		0	0	1	0				
St_{G_2}																		0	0	0	1				
$J_{\gamma_2}(1,I^{\mathrm{GL}_2}(1\otimes 1))$																						1	0	0	0
$J_{\gamma_1}(1/2,\mathrm{St}_{\mathrm{GL}_2})$																						0	1	0	0
$J_{\gamma_2}(1/2,\mathrm{St}_{\mathrm{GL}_2})$																						0	0	1	0
$\pi(1)'$																						0	0	0	1
$\pi(1)$																						0	0	0	ϱ
$I_0(G_2[1])$																						0	0	0	ε

we have the Vogan variety V_λ and the group H_λ. If ϕ is an unramified Langlands parameter with infinitesimal parameter λ, let $C_\pi \subset V_\lambda$ be the H_λ-orbit corresponding to ϕ. For use below, we define $\dim(\phi) := \dim C_\phi$ and $\dim(\pi) := \dim C_\phi$ when ϕ is the Langlands parameter for π.

Let $\Pi_\phi(G_2(F))$ be the L-packet of ϕ, whose existence was proved in [Lus95]. Since A_ϕ is both the component group of ϕ and the equivariant fundamental group of C_ϕ, the Langlands correspondence, as presented in Section 2.1, now determines a bijection

$$\mathcal{L} : \Pi_\phi(G_2(F)) \to \mathsf{Loc}_{H_\lambda}(C_\phi)^{\text{simple}}_{/\text{iso}}$$

such that $\mathcal{L}(\pi)$ corresponds to the representation $\langle\ ,\pi\rangle_\phi$ of A_ϕ. Letting ϕ run over all L-parameters with infinitesimal parameter λ, this defines a bijection

$$\mathcal{P} : \Pi_\lambda(G_2(F)) \to \mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}}$$

such that $\mathcal{P}(\pi) = \mathcal{IC}(C_\phi, \mathcal{L}(\pi))$. We make use of this bijection in Section 2.4. In Case 8, this bijection was made explicit in [CFZ].

Using the case-by-case calculations of Section 2.4, we have verified the Kazhdan-Lusztig conjecture, as stated in [CFM$^+$21, Section 10.3.3], for $G_2(F)$; this extends the result of [CFZ, Section 2.10]. We have also confirmed that the Aubert involution, as calculated in [Mui97], matches the Fourier transform of the corresponding simple perverse sheaves, using the bijection above, for all unipotent representations of $G_2(F)$, confirming the expectation of [CFM$^+$21, Section 10.3.4] and extending the result of [CFZ, Section 2.13].

2.3. **A generalization of the component group of an Arthur parameter.** We now introduce the algebraic group $\mathcal{S}^{\text{ABV}}_\phi$ promised in the introduction.

Corollary 2.1. *Let $\lambda : W_F \to {}^LG_2$ be an unramified infinitesimal parameter for $G_2(F)$. Let V_λ be the moduli space of Langlands parameters with infinitesimal parameter λ. The conormal variety Λ_λ is a bundle of prehomogeneous vector spaces over V_λ.*

Proof. Infinitesimal parameters λ are classified by by the associated prehomogeneous vector space V_λ in Proposition 1.1 and these in turn are expressed in terms of the five prehomogeneous vector spaces in Proposition 1.2. In Section 1.4 we saw that the conormal variety of each of these five prehomogeneous vector spaces all have the property that they're bundles of prehomogeneous vector spaces. □

Definition 1. Notation as above. Let x be the point for ϕ in the moduli space V, where λ is the infinitesimal parameter of ϕ. Let Λ_λ be the conormal variety and recall, by Corollary 2.1, that this is a bundle of prehomogeneous vector spaces above V_λ. Set $\mathcal{S}^{\text{ABV}}_\phi := Z_{\widehat{G_2}}(x,y)$, where $x \in V$ is a point in the moduli space V that maps to the Langlands parameter ϕ and for $y \in \Lambda^{\text{sreg}}_x$. This variety is independent of the choice of y. Also set $A^{\text{ABV}}_\phi := \pi_0(\mathcal{S}^{\text{ABV}}_\phi)$, equipped with the natural homomorphism $\mathcal{S}^{\text{ABV}}_\phi \to A^{\text{ABV}}_\phi$. It follows immediately from [CFM$^+$21, Prop. 6.1.1], ϕ is of Arthur type ψ, then $\mathcal{S}^{\text{ABV}}_\phi = \mathcal{S}_\psi$.

Definition 1 is needed to find the ABV-packet coefficients $\langle\ ,\ \rangle : \mathcal{S}^{\text{ABV}}_\phi \times \Pi^{\text{ABV}}_\phi(G_2(F)) \to \bar{\mathbb{Q}}$, which we do in the next section.

2.4. **Calculation of ABV-packet coefficients.** In this section we calculate $\pi \mapsto \langle , \pi\rangle$ for every unipotent representation π of $G_2(F)$. The results are summarized in Table 2.4.1, in which characters of A^{ABV}_ϕ are listed in the column with ϕ at the top, classified by infinitesimal parameters. We list the endoscopic groups for ϕ above ϕ. ABV-packets $\Pi^{\text{ABV}}_\phi(G_2(F))$ are found by gathering together the representations π for which the character of A^{ABV}_ϕ is non-zero.

TABLE 2.4.1. ABV-packets and ABV-packet coefficients for unipotent representations of $G_2(F)$. See Section 2.4 for how to read this table.

	λ_0	λ_1		λ_2		λ_3		λ_4				λ_5		λ_6				λ_7				λ_8			
	T	T	GL_2	T	GL_2	T	GL_2	T	GL_2	GL_2	SO_4	T	GL_2	T	GL_2	GL_2	PGL_3	T	GL_2	GL_2	G_2	T	GL_2	GL_2	PGL_3 SO_4
	ϕ_0	ϕ_{1a}	ϕ_{1b}	ϕ_{2a}	ϕ_{2b}	ϕ_{3a}	ϕ_{3b}	ϕ_{4a}	ϕ_{4b}	ϕ_{4c}	ϕ_{4d}	ϕ_{5a}	ϕ_{5b}	ϕ_{6a}	ϕ_{6b}	ϕ_{6c}	ϕ_{6d}	ϕ_{7a}	ϕ_{7b}	ϕ_{7c}	ϕ_{7d}	ϕ_{8a}	ϕ_{8b}	ϕ_{8c}	ϕ_{8d}
$I(\mu_1\nu^{a_1} \otimes \mu_2\nu^{a_2})$	1																								
$I_{\gamma_2}(a-1/2, \mu \circ \det)$		1	0																						
$I_{\gamma_2}(a-1/2, \mu \otimes \mathrm{St}_{\mathrm{GL}_2})$		0	1																						
$I_{\gamma_1}(a-1/2, \mu \circ \det)$				1	0																				
$I_{\gamma_1}(a-1/2, \mu \otimes \mathrm{St}_{\mathrm{GL}_2})$				0	1																				
$I_{\gamma_1}(1/6, \theta_3^n \circ \det)$						1	0																		
$I_{\gamma_1}(1/6, \theta_3^n \otimes \mathrm{St}_{\mathrm{GL}_2})$						0	1																		
$J_{\gamma_2}(1, I^{\mathrm{GL}_2}(1 \otimes \theta_2))$								1	0	0	0														
$J_{\gamma_1}(1/2, \theta_2 \otimes \mathrm{St}_{\mathrm{GL}_2})$								0	1	0	0														
$J_{\gamma_2}(1/2, \theta_2 \otimes \mathrm{St}_{\mathrm{GL}_2})$								0	0	1	0														
$\pi(\theta_2)$								0	0	0	1														
$I_0(G_2[-1])$								ϑ_2	ϑ_2	ϑ_2	ϑ_2														
$I_{\gamma_2}(3/2, 1_{\mathrm{GL}_2})$												1	0												
$I_{\gamma_2}(3/2, \mathrm{St}_{\mathrm{GL}_2})$												0	1												
$J_{\gamma_2}(1, I^{\mathrm{GL}_2}(\theta_3 \otimes \theta_3^{-1}))$														1	0	0	0								
$J_{\gamma_1}(1/2, \theta_3^2 \otimes \mathrm{St}_{\mathrm{GL}_2})$														0	1	0	0								
$J_{\gamma_1}(1/2, \theta_3 \otimes \mathrm{St}_{\mathrm{GL}_2})$														0	0	1	0								
$\pi(\theta_3)$														0	0	0	1								
$I_0(G_2[\theta_3])$														ϑ_3	ϑ_3	ϑ_3	ϑ_3								
$I_0(G_2[\theta_3^2])$														ϑ_3^2	ϑ_3^2	ϑ_3^2	ϑ_3^2								
1_{G_2}																		1	0	0	0				
$J_{\gamma_1}(3/2, \mathrm{St}_{\mathrm{GL}_2})$																		0	1	0	0				
$J_{\gamma_2}(5/2, \mathrm{St}_{\mathrm{GL}_2})$																		0	0	1	0				
St_{G_2}																		0	0	0	1				
$J_{\gamma_2}(1, I^{\mathrm{GL}_2}(1 \otimes 1))$																						1	0	0	0
$J_{\gamma_1}(1/2, \mathrm{St}_{\mathrm{GL}_2})$																						ϱ	1	0	0
$J_{\gamma_2}(1/2, \mathrm{St}_{\mathrm{GL}_2})$																						0	ϑ_2	1	0
$\pi(1)'$																						0	0	0	1
$\pi(1)$																						0	0	ϑ_2	ϱ
$I_0(G_2[1])$																						ε	1	ϑ_2	ε

Definition 2. Let ϕ be an unramified Langlands parameter for $G_2(F)$. Following [CFM+21], we set

$$\Pi_\phi^{\text{ABV}}(G_2(F)) := \{\pi \in \Pi_\lambda(G_2/F) \mid \mathsf{Ev}_{C_\phi}(\mathcal{P}(\pi)) \neq 0\}.$$

For $s \in \mathcal{S}_\phi^{\text{ABV}}$ and $\pi \in \Pi_\phi^{\text{ABV}}(G_2(F))$, ABV-packet coefficients are given by

$$\langle s, \pi\rangle := \operatorname{trace}_s \mathsf{NEvs}_{C_\phi} \mathcal{P}(\pi). \tag{5}$$

In this definition we make implicit use of the homomorphism $\mathcal{S}_\phi^{\text{ABV}} \to A_\phi^{\text{ABV}}$.

Let $\lambda : W_F \to \widehat{G_2}$ be an unramified infinitesimal parameter for $G_2(F)$. Then λ is determined by $\lambda(\text{Fr}) = \widehat{m}(u_1 q^{a_1}, u_2 q^{a_2})$ where u_1 and u_2 are unitary complex numbers and a_1 and a_2 are real numbers. Then

$$\begin{aligned}\lambda_0(w) &= \widehat{m}(u_1^{\text{ord}(w)}, u_2^{\text{ord}(w)}) \cdot \widehat{m}(|w|^{a_1}, |w|^{a_2})\\ &= (2\hat\gamma_1^\vee + \hat\gamma_2^\vee)(u_2^{\text{ord}(w)}|w|^{a_2}) \cdot (\hat\gamma_1^\vee + \hat\gamma_2^\vee)((u_1u_2^{-1})^{\text{ord}(w)}|w|^{a_2-a_1}).\end{aligned}$$

Let $\chi = \chi_1 \otimes \chi_2 : T \to \mathbb{C}^\times$ be the dual character of $\lambda : W_F \to \widehat{T}$. By the above form of λ, we know that $\chi_1 = \mu_2\nu^{a_2}$ and $\chi_2 = \mu_1\mu_2^{-1}\nu^{a_1-a_2}$, *i.e.*, χ is given by

$$\chi = \mu_2\nu^{a_2} \otimes \mu_1\mu_2^{-1}\nu^{a_1-a_2}, \tag{6}$$

where μ_i is the unitary unramified character of $F^\times$ defined by $\mu_i(\varpi) = u_i$.

(Case 0) Suppose $\phi : W_F' \to {}^LG_2$ is a Langlands parameter with infinitesimal parameter λ_0 corresponding to Case 0 from Proposition 1.1. Then $\lambda_0 : W_F \to \widehat{G_2}$ is an unramified infinitesimal parameter such that $\dim V_{\lambda_0} = 0$, so $\hat\gamma(\lambda(\text{Fr})) \neq q$ for every root $\hat\gamma \in \hat R$; this is Case 0 from Proposition 1.1. If we write $\lambda(\text{Fr}) = \hat m(u_1q_1^a, u_2q^{a_2})$ then this case is equivalent to the condition

$$\{u_2^2u_1^{-1}q^{2a_2-a_1}, u_1u_2^{-1}q^{a_1-a_2}, u_2q^{a_2}, u_1q^{a_1}, u_1^2u_2^{-1}q^{2a_1-a_2}, u_1u_2q^{a_1+a_2}\} \cap \{q, q^{-1}\} = \emptyset.$$

In this case H_λ is the dual of an endoscopic group for $\widehat{G_2}$; so, besides $\widehat{G_2}$ itself, H_λ is SL_3 or SO_4 or one of the Levi subgroups $\text{GL}_2^{\hat\gamma_1}$, $\text{GL}_2^{\hat\gamma_2}$ or $\widehat{T}$. For each such λ, there is a unique ϕ with $\lambda_\phi = \lambda$, *i.e.*, $\phi(w,x) = \lambda(w)$. In this case, there is a unique Langlands parameter with infinitesimal parameter λ_0 which is defined by $\phi_0(w,x) = \lambda_0(w)$. If $a_1 = a_2 = 0$, then χ_1 and χ_2 are unitary. Notice that there is a unique order 2 unramified character of $F^\times$ and thus χ_1, χ_2 cannot be two distinct order 2 unramified characters. In this case, $I(\chi_1 \otimes \chi_2)$ is irreducible by [Key82]. If one of a_1, a_2 is nonzero, then at least one of χ_1, χ_2 is non-unitary. One easily translates the condition $\hat\gamma(\lambda_0(\text{Fr})) \neq q^{\pm1}, \forall\hat\gamma \in R(\widehat{G_2}, \widehat{T})$ to the conditions on χ_1, χ_2, which are exactly the conditions of [Mui97, Proposition 3.1]. Thus, the induced representation $I(\chi_1 \otimes \chi_2)$ is also irreducible by [Mui97, Proposition 3.1]. In both cases, the unramified local Langlands correspondence says that $\pi(\phi_0) = I(\chi_1 \otimes \chi_2)$ and the L-packet is a singleton. Here, $\phi_0(w,x) = \lambda_0(w)$, given above. There are no stabilizing representations to the irreducible $I(\chi_1 \otimes \chi_2)$. This is a direct consequence of the simple observation that there is but one simple object in $\mathsf{Per}_{H_{\lambda_0}}(V_{\lambda_0})$; see also P0.

(Case 1) Suppose $\phi : W_F' \to {}^LG_2$ is a Langlands parameter with infinitesimal parameter λ_1 corresponding to Case 1 from Proposition 1.1. Then $V_{\lambda_1} = \text{Span}\{X_{\hat\gamma_1+2\hat\gamma_2}\}$ and $(2\hat\gamma_1 + 3\hat\gamma_2)(\lambda(\text{Fr})) = q$ and $\hat\gamma(\lambda(\text{Fr})) \neq q$ for every other $\hat\gamma \in \hat R$; in other words, $u_1u_2 = 1$ and $a_1 + a_2 = 1$ and

$$\{u^{-3}q^{2-3a}, u^2q^{2a-1}, u^{-1}q^{1-a}, uq^a, u^3q^{3a-1}\} \cap \{q, q^{-1}\} = \emptyset,$$

where $u := u_1$ and $a := a_1$ and $H_{\lambda_1} = \widehat{T}$ if $u^2q^{2a-1} \neq 1$ and $H_{\lambda_1} = \text{GL}_2^{\hat\gamma_1}$ if $u^2q^{2a-1} = 1$.

The action of H_{λ_1} on V_{λ_1} is given by

$$\begin{array}{rcll} \widehat{m}(x,y).rX_{\hat\gamma_1+2\hat\gamma_2} &=& xrX_{\hat\gamma_1+2\hat\gamma_2}, & \text{if } u^2q^{2a-1}\neq 1,\\ g.rX_{\hat\gamma_1+2\hat\gamma_2} &=& \det(g)rX_{\hat\gamma_1+2\hat\gamma_2}, & \text{if } u^2q^{2a-1}=1. \end{array}$$

This action has two orbits: the closed (zero) orbit and the open orbit. The corresponding Langlands parameters are explicitly given by

$$\begin{array}{rcll} \phi_{1a}(w,x) &=& \lambda_1(w), & \text{closed;}\\ \phi_{1b}(w,x) &=& \widehat{m}(1,u^{\mathrm{ord}(w)}|w|^{a-1/2})\iota_{\hat\gamma_1+2\hat\gamma_2}(x), & \text{open.} \end{array}$$

The simple objects in $\mathsf{Per}_H(V)$ are described in P1 using the base-change argument in the proof of Proposition 1.2.

By (6), the dual character of λ_1 is $\chi=\mu\nu^a\otimes\mu^{-1}\nu^{1-a}$, where $\mu : F^\times\to\mathbb{C}^1$ is the unramified unitary character determined by $\mu(\varpi)=u$. We consider the representation $I(\mu\nu^a\otimes\mu^{-1}\nu^{1-a})$. In the Grothendieck group, we have

$$\begin{aligned} I(\mu\nu^a\otimes\mu^{-1}\nu^{1-a}) &= I(\mu\nu^{a-1}\otimes\nu)\\ &= I_{\gamma_2}(a-1/2, I^{\mathrm{GL}_2}(\mu\nu^{1/2}\otimes\mu\nu^{-1/2}))\\ &= I_{\gamma_2}(a-1/2,\mu\otimes\mathrm{St}_{\mathrm{GL}_2})+I_{\gamma_2}(a-1/2,\mu\circ\det). \end{aligned}$$

By [Mui97, Theorem 3.1 (ii)], the two representations, $I_{\gamma_2}(a-1/2,\mu\otimes\mathrm{St}_{\mathrm{GL}_2})$ and $I_{\gamma_2}(a-1/2,\mu\circ\det)$, are irreducible. For these representations, the local Langlands correspondence is given by

$$\begin{array}{rcl} \Pi_{\phi_{1a}}(G_2(F)) &=& \{I_{\gamma_2}(a-1/2,\mu\circ\det)\},\\ \Pi_{\phi_{1b}}(G_2(F)) &=& \{I_{\gamma_2}(a-1/2,\mu\otimes\mathrm{St}_{\mathrm{GL}_2})\}. \end{array}$$

It follows that the equivariant perverse sheaf corresponding to $I_{\gamma_2}(a-1/2,\mu\circ\det)$ is $\mathcal{IC}(\mathbb{1}_{C_{1a}})$, where $\mathbb{1}_{C_{1a}}$ is the constant local system on $C_{1a}=\{0\}$ in $V=\mathbb{A}^1$, and the equivariant perverse sheaf corresponding to $I_{\gamma_2}(a-1/2,\mu\otimes\mathrm{St}_{\mathrm{GL}_2})$ is $\mathcal{IC}(\mathbb{1}_{C_{1b}})$, where $\mathbb{1}_{C_{1b}}$ is the constant local system on the open orbit C_{1b}. Taking the case $n=1$ of Section P1 and then using the base-change argument in the proof of Proposition 1.2, it follows that the simple objects in $\mathsf{Per}_{H_\lambda}(V_\lambda)$ are $\mathcal{IC}(\mathbb{1}_{C_{1a}})$ and $\mathcal{IC}(\mathbb{1}_{C_{1b}})$ only, and that $\mathsf{Evs}_{C_{1a}}\mathcal{IC}(\mathbb{1}_{C_{1a}})=\mathbb{1}$ and $\mathsf{Evs}_{C_{1b}}\mathcal{IC}(\mathbb{1}_{C_{1b}})=\mathbb{1}$ while $\mathsf{Evs}_{C_{1a}}\mathcal{IC}(\mathbb{1}_{C_{1b}})=0$ and $\mathsf{Evs}_{C_{1b}}\mathcal{IC}(\mathbb{1}_{C_{1a}})=0$. Therefore, there are no stabilizing representations for the L-packets above, and we have

$$\begin{array}{rcl} \Pi^{\mathrm{ABV}}_{\phi_{1a}}(G_2(F)) &=& \Pi_{\phi_{1a}}(G_2(F)),\\ \Pi^{\mathrm{ABV}}_{\phi_{1b}}(G_2(F)) &=& \Pi_{\phi_{1b}}(G_2(F)). \end{array}$$

These calculations follow from the results of Section P1.

(Case 2) Suppose ϕ is a Langlands parameter with infinitesimal parameter λ_2 from Case 2 of Proposition 1.1. This case is quite similar to the previous case. The infinitesimal parameter λ_2 is determined by $\lambda_2(\mathrm{Fr})=\widehat{m}(uq^a,u^{-1}q^{1-a})$ with $u\in\mathbb{C}^1, a\in\mathbb{R}$ satisfying ... Then $V_{\lambda_2}=\mathrm{Span}\{X_{2\hat\gamma_1+3\hat\gamma_2}\}$. and $H_\lambda=\widehat{T}$ if $u^2q^{2a-1}\neq 1$, and $H_\lambda=\mathrm{GL}_{2,\hat\gamma_2}$ if $u^2q^{2a-1}=1$. The action of H_λ on V_λ is given by

$$\begin{array}{rcll} \widehat{m}(x,y).rX_{2\hat\gamma_1+3\hat\gamma_2} &=& xyrX_{2\hat\gamma_1+3\hat\gamma_2}, & \text{if } u^2q^{2a-1}\neq 1,\\ g.rX_{2\hat\gamma_1+3\hat\gamma_2} &=& \det(g)rX_{2\hat\gamma_1+3\hat\gamma_2}, & \text{if } u^2q^{2a-1}=1. \end{array}$$

Tthis action has two orbits – the zero orbit and the open orbit – and each Langlands parameter has trivial component group. Representatives for the two L-parameters are

$$\begin{array}{rcl} \phi_{2b}(w,x) &=& \lambda_2(w)\\ \phi_{2b}(w,x) &=& \widehat{m}(u^{\mathrm{ord}(w)}|w|^{a-1/2},u^{-\mathrm{ord}(w)}|w|^{-a+1/2})\cdot\iota_{2\hat\gamma_1+3\hat\gamma_2}(x). \end{array}$$

By (6), the dual character of λ_2 is $\mu^{-1}\nu^{1-a}\otimes\mu^2\nu^{2a-1}$, where μ is the unitary unramified character of $F^\times$ determined by $\mu(\varphi)=u$. In the Grothendieck group, we have

$$\begin{aligned} I(\mu^{-1}\nu^{1-a}\otimes\mu^2\nu^{2a-1}) &= I(\mu\nu^a\otimes\mu\nu^{a-1}) \\ &= I_{\gamma_1}(a-1/2, I^{\mathrm{GL}_2}(\mu\nu^{1/2}\otimes\mu\nu^{-1/2})) \\ &= I_{\gamma_1}(a-1/2,\mu\otimes\mathrm{St}_{\mathrm{GL}_2}) + I_{\gamma_1}(a-1/2,\mu\circ\det). \end{aligned}$$

By [Mui97, Theorem 3.1 (i)], the representations $I_{\gamma_1}(a-1/2,\mu\otimes\mathrm{St}_{\mathrm{GL}_2})$ and $I_{\gamma_1}(a-1/2,\mu\cdot\det)$ are irreducible. The Langlands correspondence is given in given in Table 2.5.1. Arguing as in Case 1 or using our microlocal analysis of the prehomogeneous vector space P1, the ABV-packet coefficients are given in Table 2.4.1.

(Case 3) Suppose ϕ is an unramified Langlands parameter with infinitesimal parameter λ_3 from Case 3 of Proposition 3. We can take $\lambda(\mathrm{Fr})=\widehat{m}(\theta q^{1/3},\theta^2 q^{2/3})$ with $\theta=\theta_3^n$ for $n=0,1,2$ and $H_\lambda=\mathrm{GL}_2^{\hat\gamma_1+3\hat\gamma_2}$. To see this note that $\hat\gamma_1(\lambda(\mathrm{Fr}))=q$ and $(2\hat\gamma_1+3\hat\gamma_2)(\lambda(\mathrm{Fr}))=q$ and $\hat\gamma(\lambda(\mathrm{Fr}))\neq q$ for every other $\hat\gamma\in\hat R$, so if $\lambda(\mathrm{Fr})=\hat m(u_1q^{a_1},u_2q^{a_2})$ then $u_2=u_1^2\in\{1,\theta_3,\theta_3^2\}$ and $a_1=\frac{1}{3}$, $a_2=\frac{2}{3}$. The action of H_λ on V_λ is $\det\otimes\mathrm{Sym}^1$, which is to say, $h\cdot x=\det(h)hx$ where hx is matrix multiplication. This is case P2 for $n=1$. In particular, there are two orbits of this action: the zero orbit and the open orbit. The corresponding Langlands parameters are given by

$$\begin{aligned} \phi_{3a}(w,x) &= \lambda_3(w), \\ \phi_{3b}(w,x) &= \widehat{m}(\theta^{\mathrm{ord}(w)}|w|^{1/3},\theta^{2\,\mathrm{ord}(w)}|w|^{1/6})\cdot\iota_{\hat\gamma_1}(x). \end{aligned}$$

By (6), the dual character of λ is $\chi=\vartheta\nu^{2/3}\otimes\vartheta^2\nu^{-1/3}$ where ϑ is the unramified character of $F^\times$ dual to θ. In the Grothendieck group, we have

$$\begin{aligned} I(\theta_3^n\nu^{2/3}\otimes\theta_3^{2n}\nu^{-1/3}) &= I_{\gamma_1}(1/6, I^{\mathrm{GL}_2}(\theta_3^n\nu^{1/2}\otimes\theta_3^{-n}\nu^{-1/2})) \\ &= I_{\gamma_1}(1/6,\theta_3^n\otimes\mathrm{St}_{\mathrm{GL}_2}) + I_{\gamma_1}(1/6,\theta_3^n\circ\det). \end{aligned}$$

The Langlands correspondence is given in given in Table 2.5.1. Using our microlocal analysis of the prehomogeneous vector space P2, for $n=1$, the ABV-packet coefficients are given in Table 2.4.1.

(Case 4) Let λ_4 be the infinitesimal parameter appearing in Case 4 of Proposition 1.1. Then $\lambda_4(\mathrm{Fr})=\widehat{m}(q,-q)$ and $V_{\lambda_4}=\mathrm{Span}\{X_{\hat\gamma_1},X_{\hat\gamma_1+2\hat\gamma_2}\}$ and $H_{\lambda_4}=\widehat{T}$. The action of $\widehat{T}$ on V_{λ_4} is given by

$$\widehat{m}(x,y).(r_1X_{\hat\gamma_1}+r_2X_{\hat\gamma_1+2\hat\gamma_2}) = y^2x^{-1}r_1X_{\hat\gamma_1}+xr_2X_{\hat\gamma_1+2\hat\gamma_2}.$$

There are 4 orbits: $C_{4a}=\{0\}$, $C_{4b}=\{r_1X_{\hat\gamma_1}\mid r_1\neq 0\}$, $C_{4c}=\{r_2X_{\hat\gamma_1+2\hat\gamma_2}\mid r_2\neq 0\}$ and $C_{4d}=\{r_1X_{\hat\gamma_1}+r_2X_{\hat\gamma_1+2\hat\gamma_2}\mid r_1,r_2\neq 0\}$.

We find $\mathsf{Per}_{H_{\lambda_4}}(V_{\lambda_4})$ in Proposition 1.2. In particular, there are four H_{λ_4}-orbits in V_{λ_4}: $C_{4\mathrm{a}}=\{0\}$, $C_{4\mathrm{b}}=\{r_1X_{\hat\gamma_1}\mid r_1\neq 0\}$, $C_{4\mathrm{c}}=\{r_2X_{\hat\gamma_1+2\hat\gamma_2}\mid r_2\neq 0\}$ and $C_{4\mathrm{d}}=\overset{\circ}{V}=\{r_1X_{\hat\gamma_1}+r_2X_{\hat\gamma_1+2\hat\gamma_2}\mid r_1,r_2\neq 0\}$. Corresponding Langlands parameters are given by

$$\begin{aligned} \phi_{4\mathrm{a}}(w,x) &= \lambda(w), \\ \phi_{4\mathrm{b}}(w,x) &= \widehat{m}(|w|,(-1)^{\mathrm{ord}(w)}|w|^{1/2})\cdot\iota_{\hat\gamma_1}(x), \\ \phi_{4\mathrm{c}}(w,x) &= \widehat{m}(1,(-1)^{\mathrm{ord}(w)}|w|^{1/2})\cdot\iota_{\hat\gamma_1+2\hat\gamma_2}(x), \\ \phi_{4\mathrm{d}}(w,x) &= \widehat{m}(1,(-1)^{\mathrm{ord}(w)})\iota_{\hat\gamma_1}(x)\iota_{\hat\gamma_1+2\hat\gamma_2}(x). \end{aligned}$$

Note that we have $A_{\phi_{4d}}=\langle\hat m(1,-1)\rangle\cong\langle\theta_2\rangle$ and the component groups of the other 3 Langlands parameters are trivial.

The dual character of λ_4 is $\theta_2\nu\otimes\theta_2$ by (6). Recall that θ_2 is the unique unramified order 2 character of $F^\times$. By [Mui97, Proposition 4.1], we have

$$\begin{aligned} I(\theta_2\nu\otimes\theta_2) &= I_{\gamma_1}(1/2,\theta_2\otimes \mathrm{St}_{\mathrm{GL}_2}) + I_{\gamma_1}(1/2,\theta_2\circ\det) \\ &= I_{\gamma_2}(1/2,\theta_2\otimes \mathrm{St}_{\mathrm{GL}_2}) + I_{\gamma_2}(1/2,\theta_2\circ\det), \end{aligned} \tag{7}$$

and

$$\begin{array}{rcl} I_{\gamma_1}(1/2,\theta_2\otimes\mathrm{St}_{\mathrm{GL}_2}) &=& \pi(\theta_2)+J_{\gamma_1}(1/2,\theta_2\otimes\mathrm{St}_{\mathrm{GL}_2}),\\ I_{\gamma_2}(1/2,\theta_2\otimes\mathrm{St}_{\mathrm{GL}_2}) &=& \pi(\theta_2)+J_{\gamma_2}(1/2,\theta_2\otimes\mathrm{St}_{\mathrm{GL}_2}),\\ I_{\gamma_1}(1/2,\theta_2\circ\det) &=& J_{\gamma_2}(1,I^{\mathrm{GL}_2}(1\otimes\theta_2))+J_{\gamma_2}(1/2,\theta_2\otimes\mathrm{St}_{\mathrm{GL}_2}),\\ I_{\gamma_2}(1/2,\theta_2\circ\det) &=& J_{\gamma_2}(1,I^{\mathrm{GL}_2}(1\otimes\theta_2))+J_{\gamma_1}(1/2,\theta_2\otimes\mathrm{St}_{\mathrm{GL}_2}), \end{array} \tag{8}$$

As in [Mui97], we write $\pi(\theta_2)$ for the unique irreducible sub-representation of $I(\theta_2\nu\otimes\theta_2)$; note that $\pi(\theta_2)$ is square integrable by [Mui97, Proposition 4.1]. and all of the above equations are in the Grothendieck group of representations of $G_2(F)$.

The proof of the Langlands correspondence $\Pi_\phi(G_2(F))\to\widehat{A_\phi}$ for all unramified Langlands parameters for $G_2(F)$ with infinitesimal parameter λ_4 is similar to that of [CFZ, Theorem 2.5], which follows from some well-known facts. The Langlands correspondence is summarized in Table 2.5.1, here we only give some details of the proof of $\pi(\phi_{4\mathrm{d}},\varrho) = \mathrm{cInd}_{G_2(\mathcal{O}_F)}^{G_2(F)}(G_2[-1])$. Using [Lus84, p.270], the local system $(\phi_{4\mathrm{d}},\varrho)$ is cuspidal and thus $\pi(\phi_{4\mathrm{d}},\varrho)$ is a depth zero supercuspidal representation of the form

$$I_0(\sigma) := \mathrm{cInd}_{G_2(\mathcal{O}_F)}^{G_2(F)}(\sigma),$$

where σ is a unipotent cuspidal representation of $G_2(\mathbb{F}_q)$. There are totally 4 unipotent cuspidal representations of $G_2(\mathbb{F}_q)$ as given in [Car93, p.460]. To determine which one the representation σ is, we use the formal degree conjecture of [HII08], which is known in this case by [FOS19]. Let ψ^0 be a fixed unramified additive character of F. A simple calculation shows that the L-factor is given by

$$L(s,\phi_{4\mathrm{d}},\mathrm{Ad}) = \frac{1}{(1+q^{-2-s})(1+a^{-1-s})(1-q^{-1-s})^2},$$

and the epsilon $\epsilon(s,\phi_{4\mathrm{d}},\mathrm{Ad},\psi^0) = q^{10(1/2-s)}$. Thus

$$\gamma(0,\phi_{4\mathrm{d}},\mathrm{Ad},\psi^0) = \frac{q^9}{(q+1)(q^3+1)}.$$

The formal degree conjecture asserts that

$$\frac{\deg(\sigma)}{\mu(G_2(\mathcal{O}_F))} = \frac{\dim(\varrho)}{|A_{\phi_{4\mathrm{d}}}|}|\gamma(0,\phi_{4\mathrm{d}},\mathrm{Ad},\psi^0)|,$$

where $\mu(G_2(\mathcal{O}_F))$ is the Haar measure of $G_2(\mathcal{O}_F)$ normalized as in [HII08], *i.e.*,

$$\mu(G_2(\mathcal{O}_F)) = q^{-\dim(G_2(F))}\cdot|G_2(\mathbb{F}_q)| = q^{-8}(q^6-1)(q^2-1).$$

Thus we get

$$\deg(\sigma) = \frac{q(q^6-1)(q^2-1)}{(q^3+1)(q+1)}.$$

Comparing it with the 4 unipotent cupsidal representations of $G_2(\mathbb{F}_q)$, we get $\sigma = G_2[-1]$.

Using the microlocal analysis of the prehomogeneous vector space P3 for $n=2$ from Section 1.4, the ABV-packet coefficients are given by in Table 2.4.1.

(Case 5) We now find the ABV-packets and coefficients for Langlands parameters with infinitesimal parameter λ_5 from Case 5 of Proposition 1.1. The infinitesimal parameter λ_5 is determined by $\lambda_5(\mathrm{Fr}) = \widehat{m}(q^2, q)$. In this case, we have $V_{\lambda_5} = \mathrm{Span}\{X_{\hat\gamma_2}, X_{\hat\gamma_1+\hat\gamma_2}\}$ and $H_{\lambda_5} = \mathrm{GL}_2^{\hat\gamma_1}$. The action of H_{λ_5} on V_{λ_5} is equivalent to matrix multiplication, depending on an isomorphism of $\mathrm{GL}_2^{\hat\gamma_1}$ with GL_2. Category $\mathsf{Per}_{H_{\lambda_5}}(V_{\lambda_5}) \cong \mathsf{Per}_{\mathrm{GL}_2}(\mathbb{A}^2)$, with reference to the prehomogeneous vector space P2 for $n = 0$. In particular, this action has two orbits: the zero orbit C_{5a} and the open orbit C_{5b}. Corresponding Langlands parameters are given by

$$\begin{array}{rcl} \phi_{5a}(w,x) &=& \lambda_5(w), \\ \phi_{5b}(w,x) &=& \widehat{m}(|w|^{3/2}, |w|^{3/2}) \cdot \iota_{\hat\gamma_2}(x). \end{array}$$

By (6), the dual character of λ_5 is $\nu \otimes \nu$. In the Grothendieck group, we have

$$\begin{aligned} I(\nu\otimes\nu) &= I_{\gamma_2}(3/2, \pi(\nu^{1/2}, \nu^{-1/2})) \\ &= I_{\gamma_2}(3/2, \mathrm{St}_{\mathrm{GL}_2}) + I_{\gamma_2}(3/2, 1_{\mathrm{GL}_2}). \end{aligned}$$

The Langlands correspondence is given in given in Table 2.5.1. Using our microlocal analysis of the prehomogeneous vector space P2, for $n = 0$, the ABV-packet coefficients are given in Table 2.4.1.

(Case 6) ABV-packets and coefficients for Langlands parameters with infinitesimal parameter λ_6 is determined by $\lambda_6(\mathrm{Fr}) = \widehat{m}(\theta_3 q, \theta_3^2 q)$ from Case 6 of Proposition 1.1 are calculated as follows. In this case, we have $V_{\lambda_6} = \mathrm{Span}\{X_{\hat\gamma_1}, X_{\hat\gamma_1+3\hat\gamma_2}\}$ and $H = \widehat{T}$ and the action of $\widehat{T}$ on V_{λ_6} is given by

$$\widehat{m}(x,y).(r_1 X_{\hat\gamma_1} + r_2 X_{\hat\gamma_1+3\hat\gamma_2}) = y^2x^{-1} r_1 X_{\hat\gamma_1} + x^2 y^{-1} r_2 X_{\hat\gamma_1+3\hat\gamma_2}.$$

Using the base-change argument in the proof of Proposition 1.2 we find that $\mathsf{Per}_{H_{\lambda_6}}(V_{\lambda_6})$ is equivalent to the category of equivariant perverse sheaves on the prehomogeneous vector space P3 for $n = 3$. There are 4 orbits of this action: $C_{6a} = \{0\}, C_{6b} = \{r_1 X_{\hat\gamma_1} \mid r_1 \neq 0\}$, $C_{6c} = \{r_2 X_{\hat\gamma_1+3\hat\gamma_2}, r_2 \neq 0\}$ and $C_{6d} = \{r_1 X_{\hat\gamma_1} + r_2 X_{\hat\gamma_1+3\hat\gamma_2}, r_1, r_2 \neq 0\}$. Corresponding Langlands parameters are given by

$$\begin{array}{rcl} \phi_{6a}(w,x) &=& \lambda(w), \\ \phi_{6b}(w,x) &=& \widehat{m}(\theta_3^{\mathrm{ord}(w)}|w|, \theta_3^{2\,\mathrm{ord}(w)}|w|^{1/2}) \cdot \iota_{\hat\gamma_1}(x), \\ \phi_{6c}(w,x) &=& \widehat{m}(\theta_3^{\mathrm{ord}(w)}|w|^{1/2}, \theta_3^{2\,\mathrm{ord}(w)}|w|) \cdot \iota_{\hat\gamma_1+3\hat\gamma_2}(x), \\ \phi_{6d}(w,x) &=& \widehat{m}(\theta_3^{\mathrm{ord}(w)}, \theta_3^{2\,\mathrm{ord}(w)}) \varphi_{X_{\hat\gamma_1}+X_{\hat\gamma_1+3\hat\gamma_2}}(x), \end{array}$$

where $\varphi_{X_{\hat\gamma_1}+X_{\hat\gamma_1+3\hat\gamma_2}}(x) : \mathrm{SL}_2(\mathbb{C}) \to \widehat{G_2}$ is a group homomorphism determined by the $\mathfrak{sl}_2$-triple (e, f, h) with $e = X_{\hat\gamma_1} + X_{\hat\gamma_1+3\hat\gamma_2}$, $f = 2X_{-\hat\gamma_1} + 2X_{-(\hat\gamma_1+3\hat\gamma_2)}$ and $h = 2H_{\hat\gamma_1} + 2H_{\hat\gamma_1+3\hat\gamma_2}$. We find that $A_{\phi_{6d}}$ has order 3 and the component groups of the other 3 parameters are trivial.

The local Langlands correspondence in this case is obtained by arguments similar to the proof in case 4 and is given in Table 2.5.1. Using the microlocal analysis of the prehomogeneous vector space P3 for $n = 3$ from Section 1.4, the ABV-packet coefficients are given in Table 2.4.1.

(Case 7) Recall Case 7 from Proposition 1.1. The infinitesimal parameter λ_7 is given by $\lambda_7(w) = \widehat{m}(|w|^3, |w|^2)$. In this case, we have $V = \{r_1 X_{\hat\gamma_1} + r_2 X_{\hat\gamma_2} \mid r_1, r_2 \in \mathbb{C}\}$ and $H_{\lambda_7} = \widehat{T}$. The action of H_{λ_7} on V_{λ_7} is given by

$$\widehat{m}(x,y).(r_1 X_{\hat\gamma_1} + r_2 X_{\hat\gamma_2}) = y^2 x^{-1} r_1 X_{\hat\gamma_1} + x y^{-1} r_2 X_{\hat\gamma_2}.$$

This action has 4 orbits $C_{7a} = \{0\}$, $C_{7b} = \{r_1 X_{\hat\gamma_1}, r_1 \neq 0\}$, $C_{7c} = \{r_2 X_{\hat\gamma_2}, r_2 \neq 0\}$ and $C_{7d} = \{r_1 X_{\hat\gamma_1} + r_2 X_{\hat\gamma_2} \mid r_1, r_2 \neq 0\}$. The 4 corresponding Langlands parameters are given by

$$\begin{array}{rcl} \phi_{7a}(w,x) & = & \lambda_7(w), \\ \phi_{7b}(w,x) & = & \widehat{m}(|w|^3, |w|^{3/2}) \cdot \iota_{\hat\gamma_1}(x), \\ \phi_{7c}(w,x) & = & \widehat{m}(|w|^{5/2}, |w|^{5/2}) \cdot \iota_{\hat\gamma_2}(x), \\ \phi_{7d}(w,x) & = & \varphi_{\text{reg}}(x), \end{array}$$

where $\varphi_{\text{reg}} : \mathrm{SL}_2(\mathbb{C}) \to \widehat{G_2}$ is the group homomorphism determined by the regular orbit, *i.e.*, φ_{reg} is group homomorphism determined by an $\mathfrak{sl}_2$-triple (e,f,h) with $e = X_{\hat\gamma_1} + X_{\hat\gamma_2}$. Each Langlands parameter has trivial component group.

The dual character of λ_7 is $\nu^2 \otimes \nu$. All of the representations of with infinitesimal Langlands parameter λ_7 are components of $I(\nu^2 \otimes \nu)$. From the decomposition of $I(\nu^2 \otimes \nu)$ given in [Mui97, Proposition 4.4], we easily find the local Langlands correspondence in this case which is given in Table 2.5.1. Using P3 with $n = 1$ we find the ABV-coefficients which are given in Table 2.4.1

(Case 8) Case 8 from Proposition 1.1 was studied in [CFZ]. The Langlands correspondence is given in Table 2.5.1 and the ABV-coefficient appear as the last block of Table 2.4.1. We remark that $A^{\text{ABV}}_{\phi_{8a}} = A^{\text{ABV}}_{\phi_{8d}} = S_3$ and $A^{\text{ABV}}_{\phi_{8b}} = A^{\text{ABV}}_{\phi_{8c}} = \langle \theta_2 \rangle$.

2.5. **Fundamental properties of ABV-packet coefficients for G_2.** Let x_ϕ be the point for ϕ in the moduli space V_{λ_ϕ}; let C_ϕ be the H_{λ_ϕ}-orbit of x_ϕ in V_{λ_ϕ}. Pick $y \in V^*_{\lambda_\phi}$ such that $(x_\phi, y) \in T^*_{C_\phi}(V_{\lambda_\phi})_{\text{sreg}}$. Then

$$A^{\text{ABV}}_\phi = \pi_0(T^*_{C_\phi}(V_{\lambda_\phi}), (x_\phi, y)).$$

Let $p : T^*_{C_\phi}(V_{\lambda_\phi}) \to C_\phi$ be projection. Then p induces the group homomorphism $\pi_1(p, (x_\phi, y)) : \pi_1(T^*_{C_\phi}(V_{\lambda_\phi}), (x_\phi, y)) \to \pi_1(C_\phi, x_\phi)$ which in turn induces a group homomorphism between the equivariant quotients, $A^{\text{ABV}}_\phi \to A_\phi$. Pre-composition with this group homomorphism defines the map

$$\mathsf{Rep}(A_\phi) \to \mathsf{Rep}(A^{\text{ABV}}_\phi)$$

appearing in Theorem 2.2.

Theorem 2.2. *Let $\phi : W'_F \to {}^L G_2$ be an unramified Langlands parameter.*

(LLC) *The function $\pi \mapsto \langle\ ,\pi\rangle$ extends the local Langlands correspondence: for all unramified Langlands parameters $\phi : W'_F \to {}^L G_2$, the diagram*

$$\begin{array}{ccc} \Pi^{\text{ABV}}_\phi(G_2(F)) & \xrightarrow{\pi \mapsto \langle\ ,\pi\rangle} & \widehat{A^{\text{ABV}}_\phi} \\ \uparrow & & \uparrow \\ \Pi_\phi(G_2(F)) & \xrightarrow[\text{LLC}]{\pi \mapsto \langle\ ,\pi\rangle_\phi} & \widehat{A_\phi} \end{array} \tag{9}$$

commutes, where $\widehat{A^{\text{ABV}}_\phi}$ (resp. $\widehat{A_\phi}$) denotes the set of characters of irreducible representations of A^{ABV}_ϕ (resp. A_ϕ).

(Open) *If $\phi : W'_F \to {}^L G_2$ is open or closed then $\pi \mapsto \langle\ ,\pi\rangle$ is a bijection $\Pi^{\text{ABV}}_\phi(G_2(F)) \to \widehat{A^{\text{ABV}}_\phi}$.*

(Temp) *If ϕ is bounded upon restriction to W_F then all the representations in $\Pi^{\text{ABV}}_\phi(G_2(F))$ are tempered. If ϕ is not bounded upon restriction to W_F then $\Pi^{\text{ABV}}_\phi(G_2(F))$ may contain non-tempered representations and $\langle\ ,\ \rangle$ may not be bijective.*

(Norm) *A representation $\pi \in \Pi_\phi(G_2(F))$ is spherical if and only if ϕ is closed and $\langle\ ,\pi\rangle$ is the trivial representation of A^{ABV}_ϕ. In this case, the only ABV-packet that contains π is*

TABLE 2.5.1. L-packets and ABV-packets for all unipotent representations of $G_2(F)$, arranged by the cases appearing Proposition 1.1. Each row gives an ABV-packet, formed as the union of an L-packet and its stabilizing representations, called coronal representations in [CFM+21]. Notation and proofs are presented in Section 2.4 drawing on results from Proposition 1.2.

Langlands Parameter ϕ	L-packet $\Pi_\phi(G_2(F))$	Stabilizing (aka coronal) Representations
ϕ_0	$I(\mu_1\nu^{a_1}\otimes\mu_2\nu^{a_2})$	
ϕ_{1a}	$I_{\gamma_2}(a-1/2,\mu\circ\det)$	
ϕ_{1b}	$I_{\gamma_2}(a-1/2,\mu\otimes\mathrm{St}_{\mathrm{GL}_2})$	
ϕ_{2a}	$I_{\gamma_1}(a-1/2,\mu\circ\det)$	
ϕ_{2b}	$I_{\gamma_1}(a-1/2,\mu\otimes\mathrm{St}_{\mathrm{GL}_2})$	
$\phi_{3.a}$	$I_{\gamma_1}(1/6,\theta_3^n\circ\det)$	
$\phi_{3.b}$	$I_{\gamma_1}(1/6,\theta_3^n\otimes\mathrm{St}_{\mathrm{GL}_2})$	
ϕ_{4a}	$J_{\gamma_2}(1,I^{\mathrm{GL}_2}(1\otimes\theta_2))$	$I_0(G_2[-1])$
ϕ_{4b}	$J_{\gamma_1}(1/2,\theta_2\otimes\mathrm{St}_{\mathrm{GL}_2})$	$I_0(G_2[-1])$
ϕ_{4c}	$J_{\gamma_2}(1/2,\theta_2\otimes\mathrm{St}_{\mathrm{GL}_2})$	$I_0(G_2[-1])$
ϕ_{4d}	$\pi(\theta_2), I_0(G_2[-1])$	
ϕ_{5a}	$I_{\gamma_2}(3/2,1_{\mathrm{GL}_2})$	
ϕ_{5b}	$I_{\gamma_2}(3/2,\mathrm{St}_{\mathrm{GL}_2})$	
ϕ_{6a}	$J_{\gamma_2}(1,I^{\mathrm{GL}_2}(\theta_3\otimes\theta_3^{-1}))$	$I_0(G_2[\theta_3]), I_0(G_2[\theta_3^2])$
ϕ_{6b}	$J_{\gamma_1}(1/2,\theta_3^2\otimes\mathrm{St}_{\mathrm{GL}_2})$	$I_0(G_2[\theta_3]), I_0(G_2[\theta_3^2])$
ϕ_{6c}	$J_{\gamma_1}(1/2,\theta_3\otimes\mathrm{St}_{\mathrm{GL}_2})$	$I_0(G_2[\theta_3]), I_0(G_2[\theta_3^2])$
ϕ_{6d}	$\pi(\theta_3), I_0(G_2[\theta_3]), I_0(G_2[\theta_3^2])$	
ϕ_{7a}	1_{G_2}	
ϕ_{7b}	$J_{\gamma_1}(3/2,\mathrm{St}_{\mathrm{GL}_2})$	
ϕ_{7c}	$J_{\gamma_2}(5/2,\mathrm{St}_{\mathrm{GL}_2})$	
ϕ_{7d}	St_{G_2}	
ϕ_{8a}	$J_{\gamma_2}(1,I^{\mathrm{GL}_2}(1\otimes 1))$	$J_{\gamma_1}(1/2,\mathrm{St}_{\mathrm{GL}_2}), I_0(G_2[1])$
ϕ_{8b}	$J_{\gamma_1}(1/2,\mathrm{St}_{\mathrm{GL}_2})$	$J_{\gamma_2}(1/2,\mathrm{St}_{\mathrm{GL}_2}), I_0(G_2[1])$
ϕ_{8c}	$J_{\gamma_2}(1/2,\mathrm{St}_{\mathrm{GL}_2})$	$\pi(1), I_0(G_2[1])$
ϕ_{8d}	$\pi(1)', \pi(1), I_0(G_2[1])$	

$\Pi_\phi^{\mathrm{ABV}}(G_2(F))$. *A representation* $\pi\in\Pi_\phi(G_2(F))$ *is generic if and only if* ϕ *is open and* $\langle\ ,\pi\rangle$ *is the trivial representation of* A_ϕ^{ABV}.

(Dual) *For every unramified Langlands parameter* ϕ *there is an unramified Langlands parameter* ϕ^**, with the same infinitesimal parameter as* ϕ*, such that the Aubert involution defines a bijection*

$$\Pi_\phi^{\mathrm{ABV}}(G_2(F))\to\Pi_{\phi^*}^{\mathrm{ABV}}(G_2(F)).$$

The $Z_{\widehat{G_2}}(\lambda_\phi)$*-orbit of* ϕ *in* V_{λ_ϕ} *is dual to the* $Z_{\widehat{G_2}}(\lambda_\phi)$*-orbit of* ϕ^* *in* V_{λ_ϕ}.

Proof. All these results follow from a study of Table 2.4.1 together with results from [Mui97].

(LLC) Table 2.4.1 shows that $r:\mathsf{Rep}(A_\phi)\to\mathsf{Rep}(A_\phi^{\mathrm{ABV}})$ preserves irreducibility and is injective. It follows from [CFM+21, Theorem 7.10.1 (d)] that if $\pi\in\Pi_\phi$ then $\langle\ ,\pi\rangle = r(LLC(\pi))$ where $LLC(\pi)\in\widehat{A_\phi}$ is the representation given by the local Langlands correspondence applied to π. Alternately, this can be seen by comparing Tables 2.1.1 and 2.4.1.

(Open) In Cases 0, 1, 2, 3, 5 and 7, $\Pi_\phi^{\mathrm{ABV}}(G_2(F)) \to \widehat{A_\phi^{\mathrm{ABV}}}$ is a bijection since $\Pi_\phi^{\mathrm{ABV}}(G_2(F)) = \Pi_\phi(G_2(F))$. Observe also that every Langlands parameter is either open or closed in cases Cases 0, 1, 2, 3 and 5. Inspect Table 2.4.1 for Cases 4, 6 and 8.

(Temp) Note that when $\phi|_{W_F}$ is bounded, then ϕ is open and $\Pi_\phi^{\mathrm{ABV}}(G_2(F)) = \Pi_\phi(G_2(F))$. It follows from the previous explicit local Langlands correspondence that each $\Pi_\phi(G_2(F))$ consists of tempered representations when $\phi|_{W_F}$ is bounded.

(Norm) Recall the general fact that if $P = MN$ is a parabolic subgroup of a reductive group G over F and if σ is an irreducible representation of M, then the parabolic induction $\mathrm{Ind}_P^G(\sigma)$ has a unique spherical component if σ is spherical and has a unique generic component if σ is generic. The assertion follows from this general fact. For example, in the Case 4, from (7) and (8), we see that $\pi(\mu)$ is generic and $J_{\gamma_2}(1, \pi(1, \mu))$ is spherical. All of the other cases are similar.

(Dual) The Aubert involution is computed in [Mui97]. The involution $\phi \mapsto \phi^*$ is given by: $\phi_0^* = \phi_0$; $\phi_{1a}^* = \phi_{1b}$; $\phi_{2a}^* = \phi_{2b}$; $\phi_{3a}^* = \phi_{3b}$; $\phi_{4a}^* = \phi_{4d}$, $\phi_{4b}^* = \phi_{4c}$; $\phi_{5a}^* = \phi_{5b}$; $\phi_{6a}^* = \phi_{6d}$, $\phi_{6b}^* = \phi_{6c}$; $\phi_{7a}^* = \phi_{7d}$, $\phi_{7b}^* = \phi_{7c}$; $\phi_{8a}^* = \phi_{8d}$, $\phi_{8b}^* = \phi_{8c}$. The result now follows from Table 2.4.1. □

Remark 2.3. With reference to (Open), all unramified Langlands parameters of $G_2(F)$ are either open or closed with the exception of ϕ_{4b}, ϕ_{4c}, ϕ_{6b}, ϕ_{6c}, ϕ_{8b} and ϕ_{8c}. For all but the last two, $\pi \mapsto \langle \ , \pi\rangle$ defines a bijection $\Pi_\phi^{\mathrm{ABV}}(G_2(F)) \to \widehat{A_\phi^{\mathrm{ABV}}}$ and in these two cases, the map is surjective.

2.6. **A geometric interpretation of Muic's reducibility points.** Many results of [Mui97] on reducibility points of induced representations of G_2 can be explained by the geometry of the action of H_λ on V_λ. The following corollary is a collection of previous discussions, which explains the reducibility results of [Mui97, Proposition 3.1 and Theorem 3.1] using geometry.

Proposition 2.4. *Let $\lambda : W_F \to \widehat{T} \to \widehat{G_2}$ be an unramified infinitesimal parameter with Vogan variety V_λ and the group H_λ acting on it. Let $\chi = \chi_1 \otimes \chi_2 : T(F) \to \mathbb{C}^\times$ be the character dual to λ, which is determined by (6).*

(i) *The representation $I(\chi_1 \otimes \chi_2)$ is irreducible if and only if $V_\lambda = \{0\}$;*

(ii) *Suppose that $V_\lambda \neq \{0\}$. Then we can write $I(\chi_1 \otimes \chi_2) = I_\gamma(s, \mu \otimes \mathrm{St}_{\mathrm{GL}_2}) + I_\gamma(s, \mu \circ \det)$ for $\gamma \in \{\gamma_1, \gamma_2\}$, $s \in \mathbb{C}$ and a unitary character μ. Moreover, the two representations, $I_\gamma(s, \mu \otimes \mathrm{St}_{\mathrm{GL}_2})$ and $I_\gamma(s, \mu \circ \det)$, are irreducible if and only if the action of H_λ on V_λ has exactly two orbits.*

Proof. Item (i) is a corollary of the discussion in Case 0, Section 2.4. If $V_\lambda \neq 0$, from the above case by case study, we see that we can write $I(\chi_1 \otimes \chi_2)$ as a sum, $I_\gamma(s, \mu \otimes \mathrm{St}_{\mathrm{GL}_2}) + I_\gamma(s, \mu \circ \det)$. Note that in the cases in 1, 2, 5 and 3, the action of H_λ on V_λ has exactly two orbits, and the corresponding $I(s, \mu \otimes \mathrm{St}_{\mathrm{GL}_2})$ and $I(s, \mu \circ \det)$ are irreducible. While in the cases in subsections 4, 6, 7 and 8, the action of H_λ on V_λ has more than two orbits and the corresponding $I(s, \mu \otimes \mathrm{St}_{\mathrm{GL}_2})$ and $I(s, \mu \circ \det)$ are reducible. □

This corollary shows that how the geometry of the action of H_λ on V_λ can determine the representations, which is exactly the spirit of [CFM+21].

2.7. **Unramified Arthur parameters for G_2.**

Definition 3. Let us say that a Langlands parameter $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G_2$ *is of Arthur type* if there exists an Arthur parameter $\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G_2$, such that $\phi = \phi_\psi$, where $\phi_\psi(w, x) = \psi(w, x, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2}))$. We say that an irreducible admissible representation π of $G_2(F)$ is of Arthur type if there is a Langlands parameter ϕ of Arthur type such that $\pi \in \Pi_\phi^{\mathrm{ABV}}(G_2(F))$; if such a ϕ exists, is generally not unique.

Remark 2.5. If ψ is of Arthur type then the Arthur parameter ψ, above, is unique. If π is of Arthur type then the Langlands parameter ϕ, above, is generally not unique.

Remark 2.6. Definition 3 opens the door to a dangerous misapprehension. Every Arthur parameter ψ determines a Langlands parameter ϕ_ψ; this function from Arthur parameters to Langlands parameters is injective. We say that a Langlands parameter is of Arthur type if it lies in the image of this function. On the other hand, an irreducible admissible representation is said to be of Arthur type if it lies in an Arthur packet - typically more than one. It can happen that a representation of Arthur type may have a Langlands parameters that is not of Arthur type. An example of this phenomenon appears in [CFM+21, Section 16]. Specifically, the Arthur packet $\Pi_{\psi_2}(\mathrm{SO}_7(F))$ contains two representations, π_2^+ and π_3^-; the Langlands parameter for π_2^+ is of Arthur type ψ_2, but the Langlands parameter for π_3^- is not of Arthur type; see [CFM+21, Table 16.5.1]. Thus π_3^- is an Arthur type representations whose Langlands parameter is not of Arthur type. On the other hand, this phenomenon does not happen for any unipotent representations of $G_2(F)$ - see Corollary 2.9.

We now determine which unramified Langlands parameters for G_2 are of Arthur type.

Proposition 2.7. *Table 4.1.1 presents necessary and sufficient conditions for an unramified Langlands parameter ϕ of $G_2(F)$ to be of Arthur type.*

Proof. Note that $\psi|_{W_F\times\{1\}\times\{1\}}$ is bounded by definition of Arthur parameter. Moreover, we can write $\psi(w,x,y)=\psi(w,1,1)\psi(1,x,1)\psi(1,1,y)$. The image of $\psi(1,x,1)$ and $\psi(1,1,y)$ is either 1 or a copy of $\mathrm{SL}_2(\mathbb{C})$ in ${}^LG_2$. Observe that the images of $\psi(w,1,1)$, $\psi(1,x,1)$ and $\psi(1,1,y)$ are commutative and both $d\psi\left(1,\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),1\right)$ and $d\psi\left(1,1,\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\right)$ are in V_λ with $\lambda=\lambda_\psi$, where $d\psi$ is the differential of ψ.

(Case 0) In this case, we have $\phi_0(w,x)=\widehat{m}(u_1^{\mathrm{ord}(w)},u_2^{\mathrm{ord}(w)})\cdot\widehat{m}(|w|^{a_1},|w|^{a_2})$ with $u_1,u_2\in\mathbb{C}^1, a_1,a_2\in\mathbb{R}$ and the Vogan variety is $\{0\}$. From the above observation, $\psi(1,x,y)=1$. Thus in this case, ϕ_0 is of Arthur type if and only if $\psi_0(w,x,y)=\phi_0(w,x)$ is an Arthur parameter, which is equivalent to $a_1=a_2=0$.

(Case 1) In this case, V_{λ_1} has dimension one. Let ψ be an Arthur parameter with $\lambda_\psi=\lambda_1$. From the above observation, we get at least one of $\psi(1,x,1)$ and $\psi(1,1,y)$ is trivial. Suppose that ψ_{1b} is an Arthur parameter with $\phi_{1b}=\phi_{\psi_{1b}}$. Then $\psi_{1b}(1,x,1)=\iota_{\hat\gamma_1+2\hat\gamma_2}(x)$ and thus $\psi_{1b}(1,1,y)=1$. Now the condition $\phi_{1b}=\phi_{\psi_{1b}}$ implies that $\psi(w,1,1)=\widehat{m}(1,u^{\mathrm{ord}(w)}|w|^{a-1/2})$ with $u\in\mathbb{C}^1, a\in\mathbb{R}$, which is supposed to be bounded. Thus we get $a=1/2$. Note that, when $a=1/2$, the corresponding Arthur parameter $\psi_{1b}(w,x,y)=\phi_1(w,x)$. It is easy to see that ϕ_{1a} is of Arthur type iff $a=1/2$ and in this case the corresponding Arthur parameter is given by $\psi_{1a}(w,x,y)=\psi_{1b}(w,y,x)$.

(Case 2) This case is similar to Case 1 and thus omitted.

(Case 3) Recall that in this case, V_λ is spanned by $X_{\widehat\gamma_1}$ and $X_{2\widehat\gamma_1+3\widehat\gamma_2}$. Notice that these two roots are not orthogonal to each other. Suppose that ϕ_{3b} is of Arthur type with Arthur parameter ψ_{3b}. We have $\psi_{3b}(1,x,1)=\phi_{3b}(1,x)=\iota_{\widehat\gamma_1}(x)$. Since $\psi_{3b}(1,1,y)$ is commutative with $\iota_{\widehat\gamma_1}$ and $(d\psi_{3b})\left(1,1,\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\right)\in V_\lambda$, we must have $\psi_{3b}(1,1,y)=1$. This implies that $\psi_{3b}(w,1,1)=\phi_{3b}(w,1)$ which is unbounded. Thus there is no such Arthur parameter and ϕ_{3b} is not of Arthur type. Similarly, ϕ_{3a} is also not of Arthur type.

(Case 4) Every Langlands parameter in this case is of Arthur type and the corresponding Arthur parameter can be given explicitly,

$$\begin{aligned}
\psi_{4a}(w,x,y) &= \widehat{m}(1,(-1)^{\mathrm{ord}(w)})\iota_{\widehat{\gamma}_1}(y)\iota_{\widehat{\gamma}_1+2\widehat{\gamma}_2}(y),\\
\psi_{4b}(w,x,y) &= \widehat{m}(1,(-1)^{\mathrm{ord}(w)})\iota_{\widehat{\gamma}_1}(x)\iota_{\widehat{\gamma}_1+2\widehat{\gamma}_2}(y),\\
\psi_{4c}(w,x,y) &= \widehat{m}(1,(-1)^{\mathrm{ord}(w)})\iota_{\widehat{\gamma}_1}(y)\iota_{\widehat{\gamma}_1+2\widehat{\gamma}_2}(x),\\
\psi_{4d}(w,x,y) &= \widehat{m}(1,(-1)^{\mathrm{ord}(w)})\iota_{\widehat{\gamma}_1}(x)\iota_{\widehat{\gamma}_1+2\widehat{\gamma}_2}(x).
\end{aligned}$$

(Case 5) In this case, both parameters are not of Arthur type and the reason is the same as in the case 3.

(Case 6) The Langlands parameters ϕ_{6a} and ϕ_{6d} are of Arthur type and the corresponding Arthur parameters are given by

$$\psi_{6d}(w,x,y) = \phi_{6d}(w,x), \quad \psi_{6a}(w,x,y) = \psi_{6d}(w,y,x).$$

The parameters ϕ_{6b} and ϕ_{6c} are not of Arthur type for the following reason. Suppose that, for example, ϕ_{6c} is of Arthur type and the corresponding Arthur parameter is ψ_{6c}. Similar reason as above, $\psi_{6c}(1,1,y)$ should be commutative with $\iota_{\widehat{\gamma}_1+3\widehat{\gamma}_2}(x)$ and also $(d\psi_{6c})\left(1,1,\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\right)$ should be in V_{λ_6}. One can check that such $\psi_{6c}(1,1,y)$ does not exist.

(Case 7) The situation is similar to the above case. The parameters ϕ_{7a} and ϕ_{7d} are of Arthur type and the corresponding Arthur parameters are given by

$$\psi_{7d}(w,x,y) = \phi_{7d}(w,x), \quad \psi_{7a}(w,x,y) = \psi_{7d}(w,y,x).$$

The parameters ϕ_{7b} and ϕ_{7c} are not of Arthur type for a similar reason as in the last case.

(Case 8) All of the 4 Langlands parameters are of Arthur type as explained in [CFZ]. □

Remark 2.8. For a unipotent conjugacy class of $\widehat{G_2}$, there is a corresponding a conjugacy class of group homomorphism $\varphi : \mathrm{SL}_2(\mathbb{C}) \to \widehat{G_2}$, which determines an Arthur parameter ψ by the formula $\psi(w,x,y) = \varphi(y)$. There are totally 5 unipotent conjugacy classes of $\widehat{G_2}$, see [GG05, 2.7], for example. The Arthur parameters of these 5 unipotent conjugacy classes are: ψ_0 when $u_1 = u_2 = 1$ and $a_1 = a_2 = 0$ (corresponding to the trivial conjugacy class), ψ_{1a} when $u = 1$ (corresponding to the unipotent conjugacy class of the short root, i.e., $\mathcal{O}_{short}$ in the notation of [GG05, 2.7]), ψ_{2a} when $u = 1$ (corresponding to $\mathcal{O}_{long}$), ψ_{7a} (corresponding to $\mathcal{O}_{reg}$), and ψ_{8a} (corresponding to $\mathcal{O}_{subreg}$).

Corollary 2.9. *A unipotent representation π of $G_2(F)$ is of Arthur type if and only if its Langlands parameter is of Arthur type.*

Proof. We have found all unramified Arthur parameters in Proposition 2.7. Using Table 2.4.1, we find all unipotent representations of Arthur type. Using Table 2.1.1 we find the Langlands parameters of these representations. Using Proposition 2.7 again, we find that these Langlands parameters are all of Arthur type. □

Arthur has conjectured [Art89] that an irreducible admissible representation is unitary if and only if it is of Arthur type. In Corollary 2.10 we firm this conjecture for unipotent representations of $G_2(F)$.

Corollary 2.10. *A unipotent representation of $G_2(F)$ is unitary if and only if is of Arthur type.*

Proof. We use [Mui97] to find which unipotent representations are unitary. We use Proposition 2.7 and Table 2.4.1 to find which unipotent representations are or Arthur type. Direct inspection reveals that these two sets are the same. □

Let $\phi : W_F' \to {}^L G_2$ be a Langlands parameter for G_2 over F. We set $\dim(\phi) := \dim C_\phi$. For $\pi \in \Pi(G_2(F))$, set $\dim(\pi) := \dim(\phi_\pi)$ where ϕ_π is the Langlands parameter for π.

Proposition 2.11. *If ϕ is an unramified Langlands parameter of Arthur type ψ then $\mathcal{S}_\phi^{\text{ABV}} = \pi_0(Z_{\widehat{G}}(\psi))$ and $s_\phi := s_\psi := \psi(1,1,-1) \in \mathcal{S}_\phi^{\text{ABV}}$ has the property*

$$\langle s_\phi, \pi\rangle = (-1)^{\dim(\phi)-\dim(\pi)}\langle 1, \pi\rangle,$$

for every $\pi \in \Pi_\phi^{\text{ABV}}(G_2(F))$.

Proof. According to Definition 1, the value of $\langle s, \pi\rangle$, for any $s \in \mathcal{S}_\phi^{\text{ABV}}$, depends only on the image of s under the map $\mathcal{S}^{\text{ABV}} \to A_\phi^{\text{ABV}}$, so we identify s with its image in the following argument. The proof is based on a case by case consideration. If $A_\phi^{\text{ABV}} = 1$, this follows from that $\langle\ ,\pi\rangle \neq 0$ iff $\Pi_\phi = \{\pi\}$. The case 8 is considered in [CFZ]. Thus it suffices to consider the case (4), (6a) and (6d). We only consider the case (6a) and omit the details for the other cases. Notice that $s_{\phi_{6a}}^2 = 1$ while $A_{\phi_{6a}}^{\text{ABV}}$ has order 3 and thus $\langle s_{6a}, \pi\rangle_{\phi_{6a}} = \langle 1, \pi\rangle_{\phi_{6a}}$ for all $\pi \in \Pi_{\phi_{6a}}^{\text{ABV}}$. Thus it suffices to check that $(-1)^{\dim(\phi_{6a})-\dim(\pi)} = 1$ for all $\pi \in \Pi_{\phi_{6a}}^{\text{ABV}}(G_2(F))$. Note that $d(\phi_{6a}) = 0$ and $\dim(\pi) = 0$ or 2 by the description of $\Pi_{\phi_{6a}}^{\text{ABV}}$. The result follows. □

Remark 2.12. Clearly, Proposition 2.11 is true for many Langlands parameters that are not of Arthur type. For example, when $A_\phi^{\text{ABV}} = 1$, Proposition 2.11 is true for $s = 1$, regardless of whether ϕ is of Arthur type or not. However, there indeed exist ϕ for which $(-1)^{\dim(\phi)-\dim(\pi)}\langle 1, \pi\rangle$ does not take the form $\langle s, \pi\rangle$ for any $s \in \mathcal{S}_\phi^{\text{ABV}}$. Here is one example. Let $\phi = \phi_{6\text{b}}$ and $\pi = \text{cInd}_{G_2(\mathcal{O}_F)}^{G_2(F)}(G_2[\theta_3])$. Then $\langle\ ,\pi\rangle = \vartheta_3$ according Table 2.4.1. Note that $\dim(\phi) = 1$ and $\dim(\pi) = 2$ and thus $(-1)^{\dim(\phi)-\dim(\pi)}\langle 1, \pi\rangle = -1$. On the other hand, we have $\langle s, \pi\rangle = \vartheta_3(s) \in \left\{1, \theta_3, \theta_3^2\right\}$. Thus $(-1)^{\dim(\phi)-\dim(\pi)}\langle 1, \pi\rangle$ does not take the form $\langle s, \pi\rangle$ for any $s \in \mathcal{S}_\phi^{\text{ABV}}$, in this case. The same is true for $\phi = \phi_{6\text{c}}$ by a similar argument.

2.8. Distributions attached to ABV-packets.

Definition 4. For any Langlands parameter $\phi : W_F' \to {}^L G_2$ and any $s \in \mathcal{S}_\phi^{\text{ABV}}$ we define

$$\Theta_{\phi,s} := \sum_{\pi \in \Pi_{\phi_\psi}^{\text{ABV}}(G_2(F))} \text{trace}_s\left(\mathsf{NEvs}_{C_\phi}[\dim(\phi)]\mathcal{P}(\pi)[-\dim(\pi)]\right)\Theta_\pi. \tag{10}$$

Using Definition 1 this takes the equivalent form

$$\Theta_{\phi,s} = \sum_{\pi \in \Pi_\phi^{\text{ABV}}(G_2(F))} (-1)^{\dim(\phi)-\dim(\pi)}\ \langle s, \pi\rangle\ \Theta_\pi. \tag{11}$$

We will also use the notation $\Theta_\phi^{G_2} := \Theta_{\phi,1}^{G_2}$.

Proposition 2.13. *If ϕ is of Arthur type ψ then*

$$\Theta_{\phi,s} = \sum_{\pi \in \Pi_\phi^{\text{ABV}}(G_2(F))} \langle s_\psi\, s, \pi\rangle\ \Theta_\pi. \tag{12}$$

Proof. Using Definitions 1, 4 and Proposition 2.11, we have

$$\begin{aligned}
\Theta_{\phi,s} &= \sum_{\pi\in\Pi_\phi^{\mathrm{ABV}}(G_2(F))} \operatorname{trace}_s\left(\mathsf{NEvs}_{C_\phi}[\dim(\phi)]\mathcal{P}(\pi)[-\dim(\pi)]\right)\ \Theta_\pi \\
&= \sum_{\pi\in\Pi_\phi^{\mathrm{ABV}}(G_2(F))} \operatorname{trace}_s\left(\mathsf{NEvs}_{C_\phi}[\dim(\phi)]\mathcal{P}(\pi)[-\dim(\pi)]\right)\ \Theta_\pi \\
&= \sum_{\pi\in\Pi_\phi^{\mathrm{ABV}}(G_2(F))} (-1)^{\dim(\phi)-\dim(\pi)} \operatorname{trace}_s\left(\mathsf{NEvs}_{C_\phi}\mathcal{P}(\pi)\right)\ \Theta_\pi \\
&= \sum_{\pi\in\Pi_\phi^{\mathrm{ABV}}(G_2(F))} (-1)^{\dim(\phi)-\dim(\pi)}\langle s,\pi\rangle\ \Theta_\pi \\
&= \sum_{\pi\in\Pi_\phi^{\mathrm{ABV}}(G_2(F))} \langle s_\psi\, s,\pi\rangle\ \Theta_\pi. \qquad \square
\end{aligned}$$

The right hand side of (12) recalls the distributions used by Arthur for classical groups in [Art13] and parameters of Arthur type. As we noticed in Remark 2.12, there are Langlands parameters ϕ for which the conclusion of Proposition 2.11 does not hold. Thus Definition 4 provides a generalization of the distributions used by Arthur. In the the next result we will try to justify that it is indeed the correct generalization.

In order to state the next result, let us say that a Langlands parameter for $G_2(F)$ is elliptic if it does not factor through the L-group of Levi subgroup of $G_2(F)$. We will discuss endoscopy for $G_2(F)$ in Sections 3 and 4. We simply enumerate the elliptic Langlands parameters for $G_2(F)$ in Table 2.8.1 . The representations appearing in these L-packets are elliptic in the sense of [Art93].

TABLE 2.8.1. Elliptic representations of $G_2(F)$

L-parameter ϕ	L-packet = ABV-packet $\Pi_\phi(G_2(F)) = \Pi_\phi^{\mathrm{ABV}}(G_2(F))$	endoscopic groups	Component groups $A_\phi = A_\phi^{\mathrm{ABV}}$
ϕ_{4d}	$\pi(\theta_2), I_0(G_2[-1])$	SO_4	$\langle\theta_2\rangle$
ϕ_{6d}	$\pi(\theta_3), I_0(G_2[\theta_3]), I_0(G_2[\theta_3^2])$	PGL_3	$\langle\theta_3\rangle$
ϕ_{7d}	St_{G_2}	G_2	1
ϕ_{8d}	$\pi(1)', \pi(1), I_0(G_2[1])$	$\mathrm{SO}_4, \mathrm{PGL}_3$	S_3

Theorem 2.14. *For any unramified Langlands parameter ϕ for $G_2(F)$, recall $\mathcal{S}_\phi^{\mathrm{ABV}}$ from Definition 1 and $\Theta_{\phi,s}$ from Definition 4, for $s\in\mathcal{S}_\phi^{\mathrm{ABV}}$.*

(Basis) *The span of the distributions*

$$\Theta_{\phi,s} = \sum_{\pi\in\Pi_\phi^{\mathrm{ABV}}(G_2(F))} (-1)^{\dim(\phi)-\dim(\pi)}\langle s,\pi\rangle\ \Theta_\pi, \qquad s\in\mathcal{S}_\phi^{\mathrm{ABV}},$$

is equal to the span of the distributions Θ_π, as π ranges over $\Pi_\phi^{\mathrm{ABV}}(G_2(F))$ if and only if $\Pi_\phi^{\mathrm{ABV}}(G_2(F)) \to \widehat{A_\phi^{\mathrm{ABV}}}$ is a bijection. In this case, the inverse of the linear system of equations above is

$$\Theta_\pi = \sum_{s\in\mathcal{A}_\phi^{\mathrm{ABV}}} (-1)^{\dim(C_\phi^s)-\dim(\pi)} \frac{\overline{\langle s,\pi\rangle}}{|Z_{A_\phi^{\mathrm{ABV}}}(s)|}\ \Theta_{\phi,s}, \qquad \pi\in\Pi_\phi^{\mathrm{ABV}}(G_2(F)),$$

where ϕ is a Langlands parameter for π, the sum is taken over representatives s of the fibres of $\mathcal{S}_\phi^{\mathrm{ABV}}\to A_\phi^{\mathrm{ABV}}$ and where $\dim(C_\phi^s)$ is the dimension of the s-fixed points of the orbit $C_\phi\subseteq V_{\lambda_\phi}$. More generally, if π is a unipotent representation of $G_2(F)$ then its distribution *character Θ_π can be written as a linear combination of the distributions $\Theta_{\phi,s}$, by letting ϕ range over unramified L-parameters for $G_2(F)$ with the same infinitesimal parameter as π, and letting $s \in \mathcal{S}_\phi^{\mathrm{ABV}}$ range over a set of representatives for the fibres of $\mathcal{S}_\phi^{\mathrm{ABV}} \to A_\phi^{\mathrm{ABV}}$.*

(Stable) *Suppose Θ_ϕ is stable when ϕ is elliptic. Then Θ_ϕ is stable for all unramified ϕ. Moreover, these distributions form a basis for the space of stable unipotent distributions, letting ϕ range over unramified L-parameters for $G_2(F)$.*

Proof. Let ϕ be an unramified Langlands parameter for $G_2(F)$.

(Basis) $\Pi_\phi^{\mathrm{ABV}}(G_2(F)) \to \widehat{A_\phi^{\mathrm{ABV}}}$ is a bijection in all cases except when $\phi = \phi_{8b}$ and $\phi = \phi_{8c}$. Moreover, A_ϕ^{ABV} is trivial unless the infinitesimal parameter for ϕ is given by Cases 4, 6 or 8. Accordingly, there are precisely 10 non-trivial cases of the first claim: ϕ_{4a}, ϕ_{4b}, ϕ_{4c}, ϕ_{4d}, ϕ_{6a}, ϕ_{6b}, ϕ_{6c}, ϕ_{6d}, ϕ_{8a} and ϕ_{8d}. Consider the first of these cases, namely $\phi = \phi_{6a}$. Recall that $A_{\phi_{6a}}^{\mathrm{ABV}} = \langle\theta_3\rangle$ and let $s_3 \in \mathcal{S}_{\phi_{6a}}^{\mathrm{ABV}}$ be a representative for θ_3 above $\mathcal{S}_{\phi_{6a}}^{\mathrm{ABV}} \to A_{\phi_{6a}}^{\mathrm{ABV}}$. Then

$$\begin{aligned}
\Theta_{\phi_{6a}} &= \Theta_{\pi(\phi_{6a})} - \Theta_{\pi(\phi_{6\mathrm{d}},\vartheta_3)} - \Theta_{\pi(\phi_{6\mathrm{d}},\vartheta_3^2)}\\
\Theta_{\phi_{6a},s_3} &= \Theta_{\pi(\phi_{6a})} - \theta_3\Theta_{\pi(\phi_{6\mathrm{d}},\vartheta_3)} - \theta_3^2\Theta_{\pi(\phi_{6\mathrm{d}},\vartheta_3^2)}\\
\Theta_{\phi_{6a},s_3^2} &= \Theta_{\pi(\phi_{6a})} - \theta_3^2\Theta_{\pi(\phi_{6\mathrm{d}},\vartheta_3)} - \theta_3\Theta_{\pi(\phi_{6\mathrm{d}},\vartheta_3^2)},
\end{aligned}$$

where $\pi(\phi_{6a}) = J_{\gamma_2}(1, I^{\mathrm{GL}_2}(\theta_3 \otimes \theta_3^{-1}))$, $\pi(\phi_{6d}, \vartheta_3) = I_0(G_2[\theta_3])$ and $\pi(\phi_{6d}, \vartheta_3^2) = I_0(G_2[\theta_3^2])$. The transition matrix for this system of equations is the product of the matrix for $\langle s, \pi\rangle$, which is the character table for $A_{\phi_{6a}}^{\mathrm{ABV}} = \langle\theta_3\rangle$, and the diagonal, self-inverse matrix of signs $(-1)^{\dim(\phi)-\dim(\pi)}$:

$$\begin{pmatrix}1 & -1 & -1\\ 1 & -\theta_3 & -\theta_3^2\\ 1 & -\theta_3^2 & -\theta_3\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 1 & \theta_3 & \theta_3^2\\ 1 & \theta_3^2 & \theta_3\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

So,

$$\begin{pmatrix}1 & -1 & -1\\ 1 & -\theta_3 & -\theta_3^2\\ 1 & -\theta_3^2 & -\theta_3\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1/3 & 1/3 & 1/3\\ 1/3 & \bar\theta_3/3 & \bar\theta_3^2/3\\ 1/3 & \bar\theta_3^2/3 & \bar\theta_3/3\end{pmatrix}$$

To see the formula for the inverse given in the Theorem is correct, it only remains to verify that the diagonal entries in the first matrix are given by $(-1)^{\dim(C_\phi^s)-\dim(C_\pi)}$, which is done case-by-case. All the other case follow by the same argument. We illustrate a non-Abelian case, $\phi = \phi_{8d}$, for which $A_{\phi_{8d}}^{\mathrm{ABV}} = S_3$. Let s_2 and s_3 be representatives for the two non-trivial conjugacy classes in S_3, of orders 2 and 3, respectively. Then

$$\begin{aligned}
\Theta_{\phi_{8d}} &= \Theta_{\pi(\phi_{8d},1)} + 2\Theta_{\pi(\phi_{8d},\varrho)} + \Theta_{\pi(\phi_{8d},\varepsilon)}\\
\Theta_{\phi_{8d},s_2} &= \Theta_{\pi(\phi_{8d},1)} - \Theta_{\pi(\phi_{8d},\varepsilon)}\\
\Theta_{\phi_{8d},s_3} &= \Theta_{\pi(\phi_{8d},1)} - \Theta_{\pi(\phi_{8d},\varrho)} + \Theta_{\pi(\phi_{8d},\varepsilon)},
\end{aligned}$$

where $\pi(\phi_{8d}, 1) = \pi(1)'$, $\pi(\phi_{8d}, \varrho) = \pi(1)$ and $\pi(\phi_{8d}, \varepsilon) = I_0(G_2[1])$. The matrix for this system of equations is the character table for S_3. The two examples above illustrate that both the complex conjugation and the denominator $|Z_{A_\phi^{\mathrm{ABV}}}(s)|$ are needed for $G_2(F)$.

Note from Table 2.4.1 that $\Pi_\phi^{\mathrm{ABV}}(G_2(F)) \to \widehat{A_\phi^{\mathrm{ABV}}}$ is a bijection for all unramified Langlands parameters for $G_2(F)$ except $\phi = \phi_{8b}$ and $\phi = \phi_{8c}$. For $\phi = \phi_{8b}$ we have $A_{\phi_{8b}}^{\mathrm{ABV}} = \langle\theta_2\rangle$ and, after picking representative for θ_2 above $\mathcal{S}_{\phi_{8b}}^{\mathrm{ABV}} \to A_{\phi_{8b}}^{\mathrm{ABV}}$, we find

$$\begin{aligned}
\Theta_{\phi_{8b}} &= \Theta_{\pi(\phi_{8b})} - \Theta_{\pi(\phi_{8c})} + \Theta_{\pi(\phi_{8d},\varepsilon)}\\
\Theta_{\phi_{8b},s_2} &= \Theta_{\pi(\phi_{8b})} - \theta_2\Theta_{\pi(\phi_{8c})} + \Theta_{\pi(\phi_{8d},\varepsilon)},
\end{aligned}$$

where $\pi(\phi_{8b}) = J_{\gamma_1}(1/2, \mathrm{St}_{\mathrm{GL}_2})$, $\pi(\phi_{8c}) = J_{\gamma_2}(1/2, \mathrm{St}_{\mathrm{GL}_2})$ and, as above, $\pi(\phi_{8d}, \varepsilon) = I_0(G_2[1])$. While it is not possible to express $\Theta_{\pi(\phi_{8b})}$, $\Theta_{\pi(\phi_{8b})}$ and $\Theta_{\pi(\phi_{8b})}$ as a linear combination of $\Theta_{\phi_{8b}}$ and Θ_{ϕ_{8b},s_2}, we can write $\Theta_{\pi(\phi_{8d},\varepsilon)}$ in terms of $\Theta_{\phi_{8d}}$, Θ_{ϕ_{8d},s_2} and Θ_{ϕ_{8d},s_3}, as above, and then express $\Theta_{\pi(\phi_{8b})}$, $\Theta_{\pi(\phi_{8b})}$ and $\Theta_{\pi(\phi_{8b})}$ in terms of $\Theta_{\phi_{8b}}$, Θ_{ϕ_{8b},s_2}, $\Theta_{\phi_{8d}}$, Θ_{ϕ_{8d},s_2} and Θ_{ϕ_{8d},s_3}.

The case $\phi = \phi_{8c}$ is similar in that there are more representations in the packet $\Pi^{\mathrm{ABV}}_{\phi_{8c}}(G_2(F))$ than there are conjugacy classes in $A^{\mathrm{ABV}}_{\phi_{8c}}$, but again, we can write all the distribution characters for representations in $\Pi^{\mathrm{ABV}}_{\phi_{8c}}(G_2(F))$ using the distributions $\Theta_{\phi_{8c}}$ and Θ_{ϕ_{8c},s_2} together with the distributions $\Theta_{\phi_{8d}}$, Θ_{ϕ_{8d},s_2} and $\Theta_{\phi_{8d},3}$, as above.

(Stable) This follows from calculations that are made by working through the classification of unipotent representations of $G_2(F)$ appearing in Section 1.2. In Case 8 this is [CFZ, Theorem 2.16]. We give the details here of the proof in case 6. By [Mui97, Proposition 4.2], in the Grothendieck group of representations of $G_2(F)$, we have

$$\begin{aligned} M(\phi_{6a}) &= I(\nu\theta_3 \otimes \theta_3) = \pi(\phi_{6\mathrm{d}}, \mathbb{1}) + \pi(\phi_{6\mathrm{b}}) + \pi(\phi_{6\mathrm{c}}) + \pi(\phi_{6a}), \\ M(\phi_{6\mathrm{b}}) &= I_{\gamma_1}(1/2, \theta_3^{-1} \otimes \mathrm{St}_{\mathrm{GL}_2}) = \pi(\phi_{6\mathrm{d}}, \mathbb{1}) + \pi(\phi_{6\mathrm{b}}), \\ M(\phi_{6\mathrm{c}}) &= I_{\gamma_1}(1/2, \theta_3 \otimes \mathrm{St}_{\mathrm{GL}_2}) = \pi(\phi_{6\mathrm{d}}, \mathbb{1}) + \pi(\phi_{6\mathrm{c}}), \end{aligned}$$

where $M(\phi)$ means the standard module of $\pi(\phi)$ for a Langlands parameter ϕ. From the expression of Θ_ϕ for each ϕ appeared in the proof of last part, we easily see that

$$\begin{pmatrix} \Theta_{\phi_{6\mathrm{a}}} \\ \Theta_{\phi_{6\mathrm{b}}} \\ \Theta_{\phi_{6\mathrm{c}}} \\ \Theta_{\phi_{6\mathrm{d}}} \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Theta_{M(\phi_{6\mathrm{a}})} \\ \Theta_{M(\phi_{6\mathrm{b}})} \\ \Theta_{M(\phi_{6\mathrm{c}})} \\ \Theta_{\phi_{6\mathrm{d}}} \end{pmatrix}.$$

Note that each standard module is stable since it is a representation either induced from $\mathrm{GL}_2(F)$ or $T = \mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$. This shows that if $\Theta_{\phi_{6\mathrm{d}}}$ is stable, then each Θ_ϕ is stable for every ϕ in Case 6. For the rest cases, the proof is similar and thus omitted.

□

Remark 2.15. The appearance of $\dim(C^s_\phi)$ in Theorem 4.7 is a foreshadowing of the geometric constructions in Section 4 on geometric endoscopy.

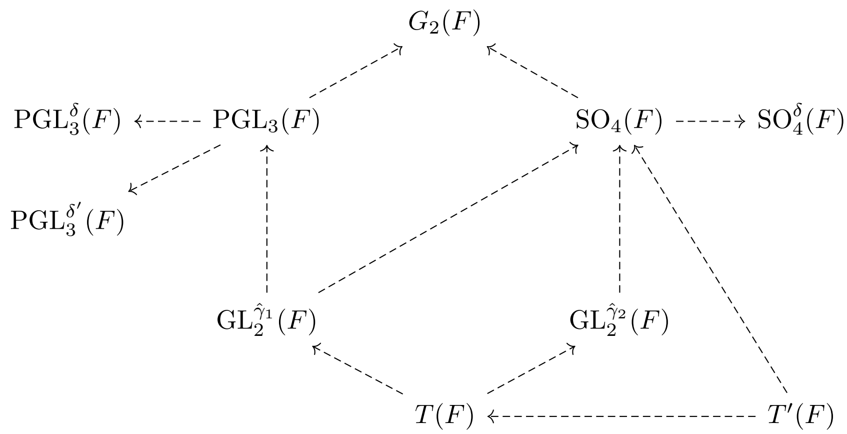
Remark 2.16. It is widely expected that Θ_ϕ is stable when ϕ is elliptic, but it seems that this has not been proved; see [CFZ, Remark 2.17]. Note that, if confirmed (Stable), could then be strengthened to: Θ_ϕ is stable for all unramified ϕ.

3. Unipotent representations of endoscopic groups for G_2

3.1. **Endoscopic groups for G_2.** Let (G, s, ξ) be an endoscopic triple for G_2. Since $s \in \widehat{G_2}$ is semisimple we choose $s \in \widehat{T}$. Since $Z_{\widehat{G_2}}(s)$ is connected, G is split over F and $\xi : {}^LG \to {}^LG_2$ is given by $\xi : \widehat{G} = Z_{\widehat{G_2}}(s) \hookrightarrow \widehat{G_2}$; see [KS99, (2.1.4a) and (2.1.4b)]. In this way we see that the endoscopic triple is determined by the pair (G, s). The endoscopic triples for G_2 are naturally arranged into the following six families, of which the last three are elliptic.

- **(T)** $G = T = \mathrm{GL}_1^2$, $s = \hat{m}(x, y)$ regular, $x^{-1}y^2 \neq 1$, $xy^{-1} \neq 1$, $x \neq 1$, $y \neq 1$, $x^2y^{-1} \neq 1$, $xy \neq 1$;
- **($\mathbf{A_1^s}$)** $G = \mathrm{GL}_2^{\hat{\gamma}_2}$, $s = \hat{m}(x, x)$, $x^2 \neq 1$;
- **($\mathbf{A_1^l}$)** $G = \mathrm{GL}_2^{\hat{\gamma}_1}$, $s = \hat{m}(x^2, x)$, $x^2 \neq 1$, $x^3 \neq 1$;
- **($\mathbf{D_2}$)** $G = \mathrm{SO}_4$ split over F, $s = \hat{m}(1, -1)$;
- **($\mathbf{A_2}$)** $G = \mathrm{PGL}_3$, $s = \hat{m}(\theta_3^2, \theta_3)$;
- **($\mathbf{G_2}$)** $G = G_2$ over F (necessarily split), $s = 1$.

TABLE 3.1.1. Endoscopic relations for G_2. The dotted arrows $G \dashrightarrow G'$ indicate that G is an endoscopic group for G' and therefore the existence of an injective admissible homomorphism $\xi : {}^LG \to {}^LG'$. There T' is the form of the torus $T = \mathrm{GL}_1^2$ that splits over the quadratic unramified extension of F.



In Sections 3.3, 3.4 and 3.5 we study the unipotent representations of GL_2, SO_4 and PGL_3 and their inner forms, respectively, including the Langlands correspondence and ABV-packets for these representations.

We will also study distributions on these groups attached to ABV-packets, for which we require the following definition, related closely to [CFM+21, Definition 3].

Let (G, s, ξ) be an endoscopic triple for G_2. Recall from [CFM+21, Section 8.1, Definition 1] that $\Pi_\phi^{\mathrm{ABV}}(G/F)$ is a subset of $\Pi^{\mathrm{pure}}(G/F)$, thus contained in the set of equivalence classes of irreducible admissible representations of $G(F)$ and all its pure inner forms. Let $\delta \in Z^1(F, G)$ be a pure inner form of G; recall that G^δ denotes form of G determined by the image of δ under $Z^1(F, G) \to Z^1(F, \mathrm{Aut}(G))$. Now let $\phi : W'_F \to {}^LG$ be an unramified Langlands parameter. With is possible that the L-packet $\Phi_\phi(G^\delta(F))$ is empty, since we have not demanded that the Langlands parameter $\phi : W'_F \to {}^LG$ is relevant to $G(F)$ in the sense of [Bor79, Section 8.3 (ii)]. If ϕ is relevant to $G^\delta(F)$, then $\Phi_\phi(G^\delta(F))$ is non-empty.

We write $\Pi_\phi^{\mathrm{ABV}}(G^\delta(F))$ for the subset of $\Pi_\phi^{\mathrm{ABV}}(G/F)$ that contains only representations of $G^\delta(F)$. Again, it is possible that $\Pi_\phi^{\mathrm{ABV}}(G^\delta(F))$ is empty when ϕ is not relevant to $G^\delta(F)$.

Definition 5. Let $\phi : W'_F \to {}^LG$ be an unramified Langlands parameter such that $s \in \mathcal{S}_{\xi\circ\phi}^{\mathrm{ABV}}$ Set

$$\Theta_{\phi,s}^{G^\delta} := e(\delta) \sum_{\pi \in \Pi_\phi^{\mathrm{ABV}}(G^\delta(F))} \mathrm{trace}_s \left(\mathsf{NEvs}_\phi[\dim(\phi)]\mathcal{P}(\pi)[-\dim(\pi)]\right)\ \Theta_\pi.$$

When written using ABV-packet coefficients for G^δ using Definition 1, the definition of $\Theta_{\phi,s}^{G^\delta}$, above, takes the form:

$$\Theta_{\phi,s}^{G^\delta} = e(\delta) \sum_{\pi \in \Pi_\phi^{\mathrm{ABV}}(G^\delta(F))} (-1)^{\dim(\phi)-\dim(\pi)}\ \langle s, \pi\rangle\ \Theta_\pi.$$

We also set $\Theta_\phi^{G^\delta} := \Theta_{\phi,1}^{G^\delta}$.

3.2. **Unipotent representations of $T = \mathrm{GL}_1^2$.** Unipotent representations of $T(F)$ are simply unramified characters $\chi_1 \otimes \chi_2 : T(F) \times \mathbb{C}^\times$. The Langlands parameter for this character is

$W_F' \to \widehat{T}$ defined by $(w,x) \mapsto \varphi_1(w) \times \varphi_2(w)$ where φ_i corresponds to χ_i under class field theory.

3.3. **Unipotent representations of** GL_2**.** The category of unipotent representations of the group $\mathrm{GL}_2(F)$ is precisely the category of unramified principle series representations of $\mathrm{GL}_2(F)$, corresponding therefore to the subcategory of smooth representations of $\mathrm{GL}_2(F)$ appearing in the Bernstein decomposition indexed by the inertial class of the trivial representation of $T(F)$. In particular, there are no supercuspidal unipotent representations of $\mathrm{GL}_2(F)$. In this section we partition the set $\Pi(\mathrm{GL}_2(F))_{\mathrm{unip}}$ of unipotent representations of $\mathrm{GL}_2(F)$ into L-packets and also describe the corresponding L-parameters, thus giving the Langlands correspondence for these representations. Since this case is very well known, we omit all proofs here.

TABLE 3.3.1. L-packets for unipotent representations of $\mathrm{GL}_2(F)$: each row in an L-packet. The notation and the corresponding L-parameters are explained below.

L-parameter ϕ	L-packet $\Pi_\phi(\mathrm{GL}_2(F))$	Arthur type?
$\phi_{3.3.0}$	$\mathrm{Ind}_{B_2(F)}^{\mathrm{GL}_2(F)}(\chi_1 \otimes \chi_2)$	unitary χ_1, χ_2
$\phi_{3.3.1a}$	$(\chi \circ \det) \otimes \mathbb{1}_{\mathrm{GL}_2}$	unitary χ
$\phi_{3.3.1b}$	$(\chi \circ \det) \otimes \mathrm{St}_{\mathrm{GL}_2}$	unitary χ

We fix the standard maximal split torus $T(F)$ of $\mathrm{GL}_2(F)$. Let $B_2(F)$ be the standard Borel subgroup of $\mathrm{GL}_2(F)$.

3.3.0 Let χ_1 and χ_2 be unramified complex characters of $\mathrm{GL}_1(F) = F^\times$ and let $\chi_1 \otimes \chi_2$ be the associated character of $T(F)$. Then the parabolically induced representation

$$I^{\mathrm{GL}_2}(\chi_1 \otimes \chi_2) := \mathrm{Ind}_{B_2(F)}^{\mathrm{GL}_2(F)}(\chi_1 \otimes \chi_2)$$

is irreducible if and only if $\chi_1\chi_2^{-1} \neq \nu^{\pm 1}$, where ν is the unramified character of $\mathrm{GL}_1(F)$ defined by $\nu(\varpi) = q$. In this case, $\mathrm{Ind}_{B_2(F)}^{\mathrm{GL}_2(F)}(\chi_1 \otimes \chi_2)$ is its own L-packet. The L-parameter $\phi : W_F' \to \widehat{\mathrm{GL}_2}$ for this irreducible representation is

$$\phi_{3.3.0}(w,x) = \begin{pmatrix} \varphi_1(w) & 0 \\ 0 & \varphi_2(w) \end{pmatrix}$$

where $\varphi_i : W_F \to \widehat{\mathrm{GL}_1}$ is the character corresponding to $\chi_i : \mathrm{GL}_1(F) \to \mathbb{C}^\times$ under local class field theory. This L-parameter is of Arthur type if and only if χ_1 and χ_2 are unitary.

3.3.1 Now suppose χ_1 and χ_2 are unramified and $\chi_1\chi_2^{-1} = \nu$. Then we may write $\chi_1 = \chi\nu^{1/2}$ and $\chi_2 = \chi\nu^{-1/2}$, in which case

$$\mathrm{Ind}_{B_2(F)}^{\mathrm{GL}_2(F)}(\chi\nu^{1/2} \otimes \chi\nu^{-1/2}) = (\chi \circ \det) \otimes \mathrm{Ind}_{B_2(F)}^{\mathrm{GL}_2(F)}(\nu^{1/2} \otimes \nu^{-1/2}).$$

The unique irreducible sub-representation of $\mathrm{Ind}_{B_2(F)}^{\mathrm{GL}_2(F)}(\nu^{1/2} \otimes \nu^{-1/2})$ is the Steinberg representation, $\mathrm{St}_{\mathrm{GL}_2}$, whereas the trivial representation, $\mathbb{1}_{\mathrm{GL}_2(F)}$, is the unique irreducible quotient. The L-parameter for $(\chi \circ \det) \otimes \mathbb{1}_{\mathrm{GL}_2(F)}$ is

$$\phi_{3.3.1a}(w,x) = \varphi(w) \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix},$$

where $\varphi : W_F \to \mathbb{C}^\times$ is the character corresponding to $\chi : F^\times \to \mathbb{C}^\times$ under class field theory. The L-parameter for $(\chi \circ \det) \otimes \mathrm{St}_{\mathrm{GL}_2}$ is

$$\phi_{3.3.1b}(w, x) = \varphi(w)x.$$

Both L-parameters appearing in this case are of Arthur type if and only if χ is unitary. We remark that $(\chi \circ \det) \otimes \mathbb{1}_{\mathrm{GL}_2(F)}$ and $(\chi \circ \det) \otimes \mathrm{St}_{\mathrm{GL}_2}$ are interchanged by Aubert-Zelevinski duality.

This completes the description of the Langlands correspondence for unipotent representations of $\mathrm{GL}_2(F)$. Each L-packet for $\mathrm{GL}_2(F)$ is its own ABV-packet.

Remark 3.1. The category of unipotent representations of $\mathrm{GL}_2(F)$ is equivalent to the category of modules over the affine Hecke algebra for $\mathrm{GL}_2(F)$ that are finite-dimensional over $\mathbb{C}$. As such, Table 3.3.1 lists the simple objects in this module category.

Theorems 2.2 and 2.14 are largely trivial in the case of $\mathrm{GL}_2(F)$ so we refrain from stating them here except to say that [CFM+21, Conjecture 1] is true for unramified representations of $\mathrm{GL}_2(F)$.

3.4. **Unipotent representations of SO_4 and its pure inner forms.** In this section we find the ABV-packets and ABV-packet coefficients for all unipotent representations of PGL_3 and its pure inner forms. The p-adic group SO_4 has two pure inner forms, both inner, since $H^1(F, \mathrm{SO}_4)$ has order 2 and

$$H^1(F, \mathrm{SO}_4) \to H^1(F, \mathrm{Aut}(\mathrm{SO}_4))$$

is injective. Let $\delta \in Z^1(F, \mathrm{SO}_4)$ be a representative for the non-trivial class in $H^1(F, \mathrm{SO}_4)$. Let SO_4^δ be the inner form of SO_4 attached to this cocycle. We partition the set $\Pi(\mathrm{SO}_4(F))_{\mathrm{unip}}$ of unipotent representations of the split orthogonal group $\mathrm{SO}_4(F)$ into L-packets and also describe the corresponding L-parameters. We do the same for the non-split inner form SO_4^δ of SO_4. The group SO_4^δ is the orthogonal group for the quadratic space coming from the norm on the unique quaternion division algebra over F.

Table 3.4.1. ABV-packets for all unipotent representations of $\mathrm{SO}_4(F)$ and its inner form $\mathrm{SO}_4^\delta(F)$

L-parameter ϕ	pure L-packet $\Pi_\phi^{\mathrm{pure}}(\mathrm{SO}_4/F)$	stabilizing representations	parameter of Arthur type?
$\phi_{3.4.0}$	$I^{\mathrm{SO}_4}(\chi_1 \otimes \chi_2)$		unitary χ_1, χ_2
$\phi_{3.4.0'}$	π_4, π_4'		yes
$\phi_{3.4.1a}$	$I_{\beta_1}(\chi \circ \det)$		unitary χ
$\phi_{3.4.1b}$	$I_{\beta_1}(\chi \otimes \mathrm{St}_{\mathrm{GL}_2})$		unitary χ
$\phi_{3.4.2a}$	$I_{\beta_2}(\chi \circ \det)$		unitary χ
$\phi_{3.4.2b}$	$I_{\beta_2}(\chi \otimes \mathrm{St}_{\mathrm{GL}_2})$		unitary χ
$\phi_{3.4.3a}$	$\chi_{\mathrm{SO}_4(F)}$	$\chi_{\mathrm{SO}_4^\delta(F)}$	yes
$\phi_{3.4.3b}$	$J_{\beta_1}(1/2, \chi \otimes \mathrm{St}_{\mathrm{GL}_2})$	$\chi_{\mathrm{SO}_4^\delta(F)}$	yes
$\phi_{3.4.3c}$	$J_{\beta_2}(1/2, \chi \otimes \mathrm{St}_{\mathrm{GL}_2})$	$\chi_{\mathrm{SO}_4^\delta(F)}$	yes
$\phi_{3.4.3d}$	$\chi_{\mathrm{SO}_4(F)} \otimes \mathrm{St}_{\mathrm{SO}_4}, \chi_{\mathrm{SO}_4^\delta(F)}$		yes

3.4.1. *Langlands correspondence.* The local Langlands correspondence for $\mathrm{SO}_4(F)$ should be well-known, but since we could not find the appropriate references, we briefly explain how the local Langlands correspondence follows from known cases.

We start with the observation that $\mathrm{GSO}_4(F) \cong (\mathrm{GL}_2(F) \times \mathrm{GL}_2(F))/\Delta(F^\times)$, see [GP92, (15.1)] for example, and therefore the local Langlands correspondence for $\mathrm{GSO}_4(F)$ follows from that of $\mathrm{GL}_2(F)$. By the general theory of [GK82] and [GT10], the local Langlands correspondence for $\mathrm{SO}_4(F)$ can be obtained in the following way. Notice that $\widehat{\mathrm{GSO}_4(F)} \cong \mathrm{GSpin}_4(\mathbb{C})$ and $\widehat{\mathrm{SO}_4(F)} \cong \mathrm{SO}_4(\mathbb{C})$. Let $\mathrm{std} : \mathrm{GSpin}_4(\mathbb{C}) \to \mathrm{SO}_4(\mathbb{C})$ be the standard homomorphism. Let $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_4(\mathbb{C})$ be a local Langlands parameter for $\mathrm{SO}_4(F)$ and let $\widetilde{\phi}$ be a local Langlands parameter for $\mathrm{GSO}_4(F)$ such that $\mathrm{std} \circ \widetilde{\phi} = \phi$ and let $\Pi_{\widetilde{\phi}}(\mathrm{GSO}_4(F))$ be the corresponding local L-packet for $\mathrm{GSO}_4(F)$, which is a singleton. Assume that $\Pi_{\widetilde{\phi}}(\mathrm{GSO}_4(F)) = \{\widetilde{\pi}\}$. Then $\widetilde{\pi}|_{\mathrm{SO}_4}$ is multiplicity free and $\Pi_\phi(\mathrm{SO}_4(F)) = \mathrm{JH}(\widetilde{\pi})$, where $\mathrm{JH}(\widetilde{\pi}) = \{\text{constituents of } \widetilde{\pi}|_{\mathrm{SO}_4(F)}\}$. Consider the subgroup $F^\times \cdot \mathrm{SO}_4(F) \subset \mathrm{GSO}_4(F)$, we have $\mathrm{GSO}_4(F)/(F^\times \cdot \mathrm{SO}_4(F)) \cong F^\times/F^{\times,2}$. By [GK82, Lemma 2.1], we have

$$\left|\Pi_\phi(\mathrm{SO}_4(F))\right| = \left|\{\chi : F^\times/F^{\times,2} \to \mathbb{C}^\times : \widetilde{\pi} \otimes \chi = \widetilde{\pi}\}\right|.$$

This allows us to find the local Langlands correspondence for unipotent representations of $\mathrm{SO}_4(F)$ explicitly, as follows.

We realize $\mathrm{GSO}_4(F)$ and $\mathrm{SO}_4(F)$ as $\mathrm{GSO}_4(F) = \{g \in \mathrm{GL}_4(F) : g^t J g = \lambda(g) J, \lambda(g) \in F^\times\}$, and $\mathrm{SO}_4(F) = \{g \in \mathrm{GSO}_4(F) : \lambda(g) = 1\}$, where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We denote the two simple roots of SO_4 (and GSO_4) by β_1 and β_2 and let $r \mapsto x_{\beta_i}(r)$ be the one parameter subgroup associated with β_i, which can be realized as

$$x_{\beta_1}(r) = \begin{pmatrix} 1 & r & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -r \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_{\beta_2}(r) = \begin{pmatrix} 1 & 0 & r & 0 \\ 0 & 1 & 0 & -r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $i = 1, 2$, the simple root β_i define an embedding $\iota_{\delta_i} : \mathrm{GL}_2(F) \to \mathrm{GSO}_4(F)$ which can be explicitly described as

$$\iota_{\beta_1}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & -c & d \end{pmatrix}, \quad x_{\beta_2}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & d \end{pmatrix}.$$

Note that $\lambda(\iota_{\beta_i}(g)) = \det(g)$ for $g \in \mathrm{GL}_2(F)$. In particular, ι_{β_i} defines an embedding $\iota_{\beta_i} : \mathrm{SL}_2(F) \to \mathrm{SO}_4(F)$. Moreover, the map ι_{β_i} define an isomorphism $\mathrm{GSO}_4(F) \cong (\mathrm{GL}_2(F) \times \mathrm{GL}_2(F))/(\Delta(F^\times))$, where $\Delta(F^\times) = \{(a, a) : a \in F^\times\} \subset \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. Thus an irreducible representation of $\mathrm{GSO}_4(F)$ is of the form $\pi_1 \boxtimes \pi_2$ with $\omega_{\pi_1} \cdot \omega_{\pi_2} = 1$, where π_i is an irreducible representation of $\mathrm{GL}_2(F)$ and ω_{π_i} is the central character of π_i. Let χ be a character of $F^\times$, we denote $I^{\mathrm{GL}_2}(\chi) = \mathrm{Ind}_{B_{\mathrm{GL}_2}}^{\mathrm{GL}_2(F)}(\chi \otimes \chi^{-1})$.

Let T_{SO_4} be the maximal torus of $\mathrm{SO}_4(F)$ which consists of elements of the form $t(a, b) := \mathrm{diag}(a, b, b^{-1}, a^{-1})$ for $a, b \in F^\times$. Let B_{SO_4} be the upper triangular Borel subgroup of $\mathrm{SO}_4(F)$. Let χ_1, χ_2 be a pair of characters of $F^\times$. We denote $\chi_1 \otimes \chi_2$ the character of T_{SO_4} defined by $\chi_1 \otimes \chi_2(t(a,b)) = \chi_1(a)\chi_2(b)$. Denote $I^{\mathrm{SO}_4}(\chi_1 \otimes \chi_2) := \mathrm{Ind}_{B_{\mathrm{SO}_4}}^{\mathrm{SO}_4(F)}(\chi_1 \otimes \chi_2)$. Note that $I^{\mathrm{SO}_4}(\chi_1 \otimes \chi_2) \cong I^{\mathrm{SO}_4}(\chi_2 \otimes \chi_1)$ in the Grothendieck group of representations of $\mathrm{SO}_4(F)$.

The group SO_4 has 2 maximal torus $P_i = M_i N_i$ with $x_{\beta_i} \in M_i$ for $i = 1, 2$, where M_i is the Levi subgroup of P_i. Note that $M_i \cong \mathrm{GL}_2(F)$. Let σ be an irreducible smooth representation of $\mathrm{GL}_2(F) \cong M_i$. Denote $I_{P_i}(\sigma) := \mathrm{Ind}_{P_i}^{\mathrm{SO}_4(F)}(\sigma)$.

The dual group $\mathrm{SO}_4(\mathbb{C})$ is realized in a similar way as $\mathrm{SO}_4(F)$ described above. In particular, if we call $\widehat{\beta}_i$ the root of $\mathrm{SO}_4(\mathbb{C})$ which is dual to β_i, then the root space $x_{\hat{\beta}_i}$ is described using the same matrix as x_{β_i}. Similarly, $\iota_{\widehat{\beta}_i}$ defines an embedding $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_4(\mathbb{C})$. Denote $\hat{t}(a,b) = \mathrm{diag}(a, b, b^{-1}, a^{-1}) \in \mathrm{SO}_4(\mathbb{C})$.

3.4.0 Let χ_1 and χ_2 be unramified characters of $F^\times$ with $\chi_1\chi_2, \chi_1\chi_2^{-1} \neq \nu^{\pm 1}$, we consider the representation $I^{\mathrm{SO}_4}(\chi_1 \otimes \chi_2)$. Then

$$I^{\mathrm{SO}_4}(\chi_1 \otimes \chi_2) = (I^{\mathrm{GL}_2}(\chi_1') \boxtimes I^{\mathrm{GL}_2}(\chi_2'))|_{\mathrm{SO}_4(F)},$$

where χ_1', χ_2' are unramified characters of $F^\times$ such that $\chi_1 = \chi_1'\chi_2', \chi_2 = \chi_2'/\chi_1'$. We remark that (χ_1', χ_2') are uniquely determined by (χ_1, χ_2) up to a twist by θ_2, where θ_2 is the unique unramified quadratic character of $F^\times$. There are two cases to consider.

(i) Suppose $(\chi_1, \chi_2) \neq (\theta_2, 1)$ and $(\chi_1, \chi_2) \neq (1, \theta_2)$. Then the representation $I^{\mathrm{SO}_4}(\chi_1 \otimes \chi_2)$ is irreducible. The Langlands parameter is given by

$$\phi_{3.4.0}(w, x) = \hat{t}(\varphi_1(w), \varphi_2(w)),$$

where $\varphi_i : W_F \to \mathbb{C}^\times$ is the character dual to χ_i, and

$$\Pi_{\phi_{3.4.0}}(\mathrm{SO}_4(F)) = \{I^{\mathrm{SO}_4}(\chi_1 \otimes \chi_2)\}.$$

The moduli space of Langlands parameters with the infinitesimal parameter $\lambda_{3.4.0}$ for $\phi_{3.4.0}$ is P0: $V_{\lambda_{3.4.0}} = \{0\}$ and where $H_{\lambda_{3.4.0}}$ is equal to: GL_2, when exactly one of $\varphi_1\varphi_2, \varphi_1\varphi_2^{-1}$ is 1; SO_4, when $\varphi_1\varphi_2 = \varphi_1\varphi_2^{-1} = 1$; $\widehat{T} = \mathrm{GL}_1^2$ in all other cases where $(\varphi_1, \varphi_2) \neq (\theta_2, 1)$ and $(\varphi_1, \varphi_2) \neq (1, \theta_2)$.

(ii) Suppose $(\chi_1, \chi_2) = (\theta_2, 1)$ or $(\chi_1, \chi_2) = (1, \theta_2)$. Then

$$I^{\mathrm{SO}_4}(\theta_2 \otimes 1) = I^{\mathrm{SO}_4}(1 \otimes \theta_2)$$

and this representation is a direct sum of two irreducible tempered representations of $\mathrm{SO}_4(F)$. We denote these two representations by π_4 and π_4'. Only one of π_4, π_4' is $\mathrm{SO}_4(\mathcal{O}_F)$-spherical, *i.e.*, has a nonzero vector fixed by the hyperspecial group $\mathrm{SO}_4(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers of F, and we choose notation so that π_4 is $\mathrm{SO}_4(\mathcal{O}_F)$-spherical. The Langlands parameter for π_4 and π_4' is

$$\phi_{3.4.0'}(w, x) = \hat{t}(\varphi_1(w), \varphi_2(w)),$$

where φ_i corresponds to χ_i under class field theory. and

$$\Pi_{\phi_{3.4.0'}}(\mathrm{SO}_4(F)) = \{\pi_4, \pi_4'\}.$$

The moduli space of Langlands parameters with the infinitesimal parameter $\lambda_{3.4.0'}$ for $\phi_{3.4.0'}$ is P0: $V_{\lambda_{3.4.0'}} = \{0\}$ where $H_{\lambda_{3.4.0}}$ is the disconnected group $\mathrm{S}(\mathrm{O}_2 \times \mathrm{O}_2)$, namely, the group generated by $\widehat{T}$ and J.

3.4.1 Let χ be an unramified character of $F^\times$ with $\chi^2 \neq \nu^{\pm 1}$. The irreducible components of $I(\chi\nu^{1/2} \otimes \chi\nu^{-1/2})$ are $I_{\beta_1}(\chi \otimes \mathrm{St}_{\mathrm{GL}_2})$ and $I_{\beta_1}(\chi \circ \det)$, where $I_{\beta_1}(\chi \otimes \mathrm{St}_{\mathrm{GL}_2}) = (\mathrm{St}_{\mathrm{GL}_2} \boxtimes I^{\mathrm{GL}_2}(\chi))|_{\mathrm{SO}_4(F)}$ and $I_{\beta_1}(\chi \circ \det) = (1_{\mathrm{GL}_2} \boxtimes I^{\mathrm{GL}_2}(\chi))|_{\mathrm{SO}_4(F)}$. The local Langlands parameter of $I_{\beta_1}(\chi \circ \det)$ is given by

$$\phi_{3.4.1a}(w, x) = \hat{t}(\varphi(w)|w|^{1/2}, \varphi(w)|w|^{-1/2}),$$

and the Langlands parameter of $I_{\beta_1}(\chi\otimes \mathrm{St}_{\mathrm{GL}_2})$ is given by

$$\phi_{3.4.1b}(w,x)\to \hat{t}(\varphi(w),\varphi(w))\iota_{\widehat{\beta}_1}(x),$$

where $\varphi : W_F\to\mathbb{C}^\times$ is the dual character of $\chi : F^\times\to\mathbb{C}^\times$. The Vogan variety $V_{\lambda_{3.4.1}}$ is the root space $\mathfrak{u}_{\hat{\beta}_1}$ with $H_{\lambda_{3.4.1}}=\widehat{T}$ action given by $t\cdot x=\beta_1(t)x$. This is equivalent to P1, and the two orbits in this prehomogenous vector space, C_0 and C_1, match $\phi_{3.4.1a}$ and $\phi_{3.4.1b}$, respectively.

3.4.2 Let χ be an unramified character with $\chi^2\neq\nu^{\pm1}$. The irreducible components of $I(\chi\nu^{1/2}\otimes\chi^{-1}\nu^{1/2})$ are $I_{\beta_2}(\chi\otimes\mathrm{St}_{\mathrm{GL}_2})$ and $I_{\beta_2}(\chi\circ\det)$ where $I_{\beta_2}(\chi\otimes\mathrm{St}_{\mathrm{GL}_2})=(I^{\mathrm{GL}_2}(\chi)\boxtimes\mathrm{St}_{\mathrm{GL}_2})|_{\mathrm{SO}_4(F)}$ and $I_{\beta_2}(\chi\circ\det)=(I^{\mathrm{GL}_2}(\chi)\boxtimes 1_{\mathrm{GL}_2})|_{\mathrm{SO}_4(F)}$. The local Langlands parameter for $I_{\beta_2}(\chi\circ\det)$ is

$$\phi_{3.4.2a}\,(w,x)=\hat{t}(\varphi(w)|w|^{1/2},\varphi(w)^{-1}|w|^{1/2}),$$

and the Langlands parameter for $I_{\beta_2}(\chi\otimes\mathrm{St}_{\mathrm{GL}_2})$ is

$$\phi_{3.4.2b}\left(w,\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)=\hat{t}(\varphi(w),\varphi(w)^{-1})\iota_{\widehat{\beta}_2}(x),$$

where $\varphi : W_F\to\mathbb{C}^\times$ is the dual character of χ. The Vogan variety $V_{\lambda_{3.4.2}}$ is the root space $\mathfrak{u}_{\hat{\beta}_2}$ with $H_{\lambda_{3.4.2}}=\widehat{T}$ action given by $t\cdot x=\widehat{\beta}_2(t)x$. This is equivalent to P1, and the two orbits in this prehomogenous vector space, C_0 and C_1, match $\phi_{3.4.2a}$ and $\phi_{3.4.2b}$, respectively.

3.4.3 Let χ be an unramified character of $F^\times$ such that $\chi^2=1$. Thus χ is either trivial or θ_2. The representation $I(\chi\nu\otimes\chi)$ has 4 irreducible components:

$$\chi_{\mathrm{SO}_4(F)},\ J_{\beta_1}(1/2,\chi\otimes\mathrm{St}_{\mathrm{GL}_2}),\ J_{\beta_2}(1/2,\chi\otimes\mathrm{St}_{\mathrm{GL}_2}),\ \chi_{\mathrm{SO}_4(F)}\otimes\mathrm{St}_{\mathrm{SO}_4},$$

where χ_{SO_4} is the character χ composed with the spinor norm of $\mathrm{SO}_4(F)$, $\mathrm{St}_{\mathrm{SO}_4}$ is the Steinberg representation of $\mathrm{SO}_4(F)$, which is the unique subrepresentation of $I^{\mathrm{SO}_4}(\nu\otimes 1)$, and $J_{\beta_i}(s,\chi\circ\mathrm{St}_{\mathrm{GL}_2})$ is the unique quotient of $I_{\beta_i}(\nu^s\chi\otimes\mathrm{St}_{\mathrm{GL}_2})$ for $s>0$. These 4 representations can also expressed as

$$\begin{aligned}\chi_{\mathrm{SO}_4}&=(\chi\circ\det\boxtimes 1_{\mathrm{GL}_2})|_{\mathrm{SO}_4}, & J_{\beta_1}(1/2,\chi\otimes\mathrm{St}_{\mathrm{GL}_2})&=(\mathrm{St}_{\mathrm{GL}_2}\boxtimes\chi\circ\det)|_{\mathrm{SO}_4(F)},\\ \mathrm{St}_{\mathrm{SO}_4}&=(\mathrm{St}_{\mathrm{GL}_2}\boxtimes\mathrm{St}_{\mathrm{GL}_2})|_{\mathrm{SO}_4(F)}, & J_{\beta_2}(1/2,\chi\otimes\mathrm{St}_{\mathrm{GL}_2})&=(\chi\circ\det\boxtimes\mathrm{St}_{\mathrm{GL}_2})|_{\mathrm{SO}_4}.\end{aligned}$$

The local Langlands parameters of these 4 representations are given, in order, by

$$\begin{aligned}\phi_{3.4.3a}(w,x)&=\hat{t}(\varphi(w)|w|,\varphi(w)),\\ \phi_{3.4.3b}(w,x)&=\hat{t}(\varphi(w)|w|^{1/2},\varphi(w)|w|^{1/2})\iota_{\widehat{\beta}_1}(x),\\ \phi_{3.4.3c}(w,x)&=\hat{t}(\varphi(w)|w|^{1/2},\varphi(w)|w|^{-1/2})\iota_{\widehat{\beta}_2}(x),\\ \phi_{3.4.3d}(w,x)&=\hat{t}(\varphi(w),\varphi(w))\iota_{\widehat{\beta}_1}(x)\iota_{\widehat{\beta}_2}(x).\end{aligned}$$

Here $\varphi : W_F\to\mathbb{C}^\times$ is the dual character of the quadratic character $\chi : F^\times\to\mathbb{C}^\times$. Let $\lambda_{3.4.3} : W_F\to {}^L\mathrm{SO}_4$ be the infinitesimal parameter for any of these Langlands parameters; thus,

$$\lambda_{3.4.3}(w)=\hat{t}(\varphi(w)|w|,\varphi(w)).$$

The moduli space of Langlands parameters with infinitesimal parameter $\lambda_{(3.4.3)}$ is the direct sum $\mathfrak{u}_{\hat{\beta}_1}\oplus\mathfrak{u}_{\hat{\beta}_2}$ of the root space for the simple roots $\hat{\beta}_1$ and $\hat{\beta}_2$ equipped with the group action of $\widehat{T}$ given by $t\cdot(x_1,x_2):=(\hat{\beta}_1(t)x_1,\hat{\beta}_2(t)x_2)$. As a prehomogeneous vector space, $V_{\lambda_{(3.4.3)}}=\mathbb{A}^2$ with this $H_{\lambda_{(3.4.3)}}=\mathrm{GL}_1^2$ action is equivalent to P3 for $n=2$.

3.4.2. *ABV-packets.* Table 3.4.1 presents the ABV-packets for all unipotent representations of SO_4 and its pure inner form SO_4^δ, each as a union of a pure L-packet and its stabilizing representations.

We now sketch how the calculations in Table 3.4.2 are made; the arguments are very similar to those in Section 3.5.2 so we will be brief here. The bijection between $\Pi^{\mathrm{pure}}_{\lambda_{3.4.3}}(\mathrm{SO}_4/F)$ and the six simple perverse sheaves in $\mathsf{Per}_{\widehat{T}}(\mathfrak{u}_{\hat\beta_1}\oplus\mathfrak{u}_{\hat\beta_2})$ is given by the following table.

	$C_{3.4.3a}$	$C_{3.4.3b}$	$C_{3.4.3c}$	$C_{3.4.3d}$
$\chi_{\mathrm{SO}_4(F)}$	$\mathcal{IC}(\mathbb{1}_{C_{3.4.3a}})$			
$J_{\beta_1}(1/2,\chi\otimes\mathrm{St}_{\mathrm{GL}_2})$		$\mathcal{IC}(\mathbb{1}_{C_{3.4.3b}})$		
$J_{\beta_2}(1/2,\chi\otimes\mathrm{St}_{\mathrm{GL}_2})$			$\mathcal{IC}(\mathbb{1}_{C_{3.4.3c}})$	
$\chi_{\mathrm{SO}_4(F)}\otimes\mathrm{St}_{\mathrm{SO}_4}$				$\mathcal{IC}(\mathbb{1}_{C_{3.4.3d}})$
$\chi_{\mathrm{SO}_4^\delta(F)}$				$\mathcal{IC}(\mathcal{D}_{C_{3.4.3d}})$

where $\mathcal{D}$ is the character of $A_{\phi_{3.4.3}}=\{\pm1\}$ corresponding to the pure inner form $\delta\in Z^1(F,\mathrm{SO}_4)$.

To illustrate the calculations that this table summarizes, consider the case $\phi_{3.4.3}$. The component group of $\phi_{(3.4.3)}$ is $\{\pm1\}$ and thus the pure L-packet of $\phi_{(3.4.3)}$ has 2 elements:

$$\Pi^{\mathrm{pure}}_{\phi_{3.4.3}}(\mathrm{SO}_4)=\{\mathrm{St}_{\mathrm{SO}_4}(\chi),\chi_{\mathrm{SO}_4^\delta(F)}\}.$$

where $\chi_{\mathrm{SO}_4^\delta(F)}=1$ if $\chi=1$ and $\chi_{\mathrm{SO}_4^\delta(F)}$ is the unique unramified quadratic character of $\mathrm{SO}_4^\delta(F)$ if $\chi\neq1$. Table 3.4.2 presents the ABV-packet coefficients promised in the Introduction as it pertains to the Langlands parameter $\phi_{3.4.3}$. From Table 3.4.2 we read off the ABV-packets in Case (3.4.3); the result appears in Table 3.4.1.

TABLE 3.4.2. ABV-packet coefficients $\langle\ ,\ \rangle$ for unipotent representations of SO_4 with infinitesimal parameter $\lambda_{3.4.3}$. The first four rows refer to representations of $\mathrm{SO}_4(F)$ while the fifth row refers to a representation of the inner form $\mathrm{SO}_4^\delta(F)$.

$\Pi^{\mathrm{pure}}_{\lambda_{3.4.3}}(\mathrm{SO}_4/F)$	$\widehat{A^{\mathrm{ABV}}_{\phi_{3.4.3a}}}$	$\widehat{A^{\mathrm{ABV}}_{\phi_{3.4.3b}}}$	$\widehat{A^{\mathrm{ABV}}_{\phi_{3.4.3c}}}$	$\widehat{A^{\mathrm{ABV}}_{\phi_{3.4.3d}}}$
$\chi_{\mathrm{SO}_4(F)}$	$\mathbb{1}$	0	0	0
$J_{\beta_1}(1/2,\chi\otimes\mathrm{St}_{\mathrm{GL}_2})$	0	$\mathbb{1}$	0	0
$J_{\beta_2}(1/2,\chi\otimes\mathrm{St}_{\mathrm{GL}_2})$	0	0	$\mathbb{1}$	0
$\mathrm{St}_{\mathrm{SO}_4}(\chi)$	0	0	0	$\mathbb{1}$
$\chi_{\mathrm{SO}_4^\delta(F)}$	δ	δ	δ	δ

Properties (LLC), (Open), (Temp) and (Norm) are elementary for $\mathrm{SO}_4(F)$ and its pure inner forms. Proposition (Stable) is interesting for $\mathrm{SO}_4(F)$ and its pure inner forms.

Proposition 3.2. *The distributions $\Theta^{\mathrm{SO}_4}_\phi$ (resp. $\Theta^{\mathrm{SO}_4^\delta}_\phi$) form a basis for the space of invariant distributions spanned by characters of unipotent irreducible admissible representations π of $\mathrm{SO}_4(F)$ (resp. $\mathrm{SO}_4^\delta(F)$). The distributions $\Theta^{\mathrm{SO}_4}_\phi$ and $\Theta^{\mathrm{SO}_4^\delta}_\phi$ are stable.*

Proof. In all cases except $\phi=\phi_{3.4.0'}$, $\Pi_\phi(\mathrm{SO}_4(F))$ and $\Pi_\phi(\mathrm{SO}_4^\delta(F))$ are singletons (or empty) and $\Theta^{\mathrm{SO}_4}_\phi=\Theta_\pi$. To see that these are all stable distributions, we may argue as follows.

In Case 3.4.0, the representations $I^{\mathrm{SO}_4}(\chi_1\otimes\chi_2)$ are stable because they are standard modules, stably induced from representations of $T(F)$.

In Case 3.4.1 we have $\Theta_{I^{\mathrm{SO}_4}(\chi\nu^{1/2}\otimes\chi\nu^{-1/2})}=\Theta_{I_{P_1}(\chi\otimes\mathrm{St}_{\mathrm{GL}_2})}+\Theta_{I_{P_1}(\chi\circ\det)}$.
Since the distribution $\Theta_{I^{\mathrm{SO}_4}(\chi\nu^{1/2}\otimes\chi\nu^{-1/2})}$ is stable (by an argument as above) and $\Theta_{I_{P_1}(\chi\circ\det)}$ is

stable by hand, $\Theta_{I_{P_1}(\chi\circ\mathrm{St}_{\mathrm{GL}_2})}$ is also stable. For Case 3.4.2 argue as in Case 3.4.1. In Case 3.4.3 we have

$$\Theta_{I^{\mathrm{SO}_4}(\chi\nu\otimes\chi)} = \Theta_{J_\emptyset(\chi\nu\otimes\chi)} + \Theta_{J_{P_1}(1/2,\chi\otimes\mathrm{St}_{\mathrm{GL}_2})} + \Theta_{J_{P_2}(1/2,\chi\otimes\mathrm{St}_{\mathrm{GL}_2})} + \Theta_{\mathrm{St}_{\mathrm{SO}_4}(\chi)}.$$

Since $I^{\mathrm{SO}_4}(\chi\nu\otimes\chi)$ is stable (argue as in Case 3.4.0) and $J_\emptyset(\chi\nu\otimes\chi)$, $J_{P_1}(1/2,\chi\otimes\mathrm{St}_{\mathrm{GL}_2})$, and $J_{P_2}(1/2,\chi\otimes\mathrm{St}_{\mathrm{GL}_2})$ are stable (argue as in Case 3.4.1) it follows that $\Theta_{\mathrm{St}_{\mathrm{SO}_4}(\chi)}$ is stable. Finally, $\chi_{\mathrm{SO}_4^\delta(F)}$ is stable because $\mathbb{1}_{\mathrm{SO}_4^\delta(F)}$ is stable. Case $\phi=\phi_{3.4.0'}$ is a well known: $\Pi_{\phi_{3.4.0'}}(\mathrm{SO}_4(F)) = \{\pi_4,\pi_4'\}$ where π_4 and π_4' are the two irreducible representations appearing in the induced representation $I^{\mathrm{SO}_4}(\chi_2\otimes 1)$ and the stable distribution attached to this L-packet is $\Theta^{\mathrm{SO}_4}_{\phi_{3.4.0'}} = \Theta_{\pi_4}+\Theta_{\pi_4'}$. □

3.5. **Unipotent representations of** PGL_3 **and its pure inner forms.** In this section we find the ABV-packets and ABV-packet coefficients for all unipotent representations of PGL_3 and its pure inner forms.

As in Section 3.3, the category of unipotent representations of $\mathrm{PGL}_3(F)$ is precisely the category of unramified principle series representations of $\mathrm{PGL}_3(F)$. In this section we partition the set $\Pi(\mathrm{PGL}_3(F))_{\mathrm{unip}}$ of unipotent representations of $\mathrm{PGL}_3(F)$ into L-packets and also describe the corresponding L-parameters. We do the same for the non-split inner form $D^\times/F^\times$, where D is a central division algebra of degree 3 over F. The p-adic group PGL_3 has three pure inner forms but only two inner forms, since $H^1(F,\mathrm{PGL}_3)$ has order 3, while the image of

$$H^1(F,\mathrm{PGL}_3)\to H^1(F,\mathrm{Aut}(\mathrm{PGL}_3))$$

has order 2. The function above is neither injective nor surjective. Let $\delta,\delta'\in Z^1(F,\mathrm{PGL}_3)$ be representatives for the two non-trivial classes in $H^1(F,\mathrm{PGL}_3)$. Let $\mathrm{PGL}_3^\delta,\mathrm{PGL}_3^{\delta'}$ be the inner forms of PGL_3 attached to these cocycles; then $\mathrm{PGL}_3^\delta\cong\mathrm{PGL}_3^{\delta'}$ as algebraic groups over F, although $[\delta]\neq[\delta']$ in $H^1(F,\mathrm{PGL}_3)$. We can take $\mathrm{PGL}_3^\delta(F)=D^\times/F^\times$ and $\mathrm{PGL}_3^{\delta'}(F)=(D')^\times/F^\times$, where D' is the opposite algebra of D.

TABLE 3.5.1. Pure Arthur packets for unipotent representations of $\mathrm{PGL}_3(F)$ and its pure inner forms PGL_3^δ and $\mathrm{PGL}_3^{\delta'}$: each row gives a pure L-packet for an L-parameter and its stabilizing representations and therefore determines a pure Arthur packet. Notation is explained below. Cases, indicated in the left-hand column, gather representations together by infinitesimal parameter.

L-parameter ϕ	pure L-packet $\Pi_\phi^{\mathrm{pure}}(\mathrm{PGL}_3\,/F)$	stabilizing representations	parameter of Arthur type?
$\phi_{3.5.0}$	$\mathrm{Ind}_{B_3(F)}^{\mathrm{GL}_3(F)}(\chi_1\otimes\chi_2\otimes\chi_1^{-1}\chi_2^{-1})$		unitary χ_1, χ_2
$\phi_{3.5.1a}$	$I_{\alpha_1}((\chi\circ\det)\otimes\chi^{-2})$		unitary χ
$\phi_{3.5.1b}$	$I_{\alpha_1}(\chi\mathrm{St}_{\mathrm{GL}_2}\otimes\chi^{-2})$		unitary χ
$\phi_{3.5.2a}$	$I_{\alpha_1}((\chi\nu^{1/6}\circ\det)\otimes\chi\nu^{-1/3})$		no
$\phi_{3.5.2b}$	$I_{\alpha_1}(\chi\nu^{1/6}\mathrm{St}_{\mathrm{GL}_2}\otimes\chi\nu^{-1/3})$		no
$\phi_{3.5.3a}$	χ_{PGL_3}	$\chi_{\mathrm{PGL}_3^\delta(F)}$, $\chi_{\mathrm{PGL}_3^{\delta'}(F)}$	yes
$\phi_{3.5.3b}$	$\chi\otimes J_{\alpha_1}(\nu\otimes\nu^{-1/2}\mathrm{St}_{\mathrm{GL}_2})$	$\chi_{\mathrm{PGL}_3^\delta(F)}$, $\chi_{\mathrm{PGL}_3^{\delta'}(F)}$	no
$\phi_{3.5.3c}$	$\chi\otimes J_{\alpha_1}(\nu^{1/2}\mathrm{St}_{\mathrm{GL}_2}\otimes\nu^{-1})$	$\chi_{\mathrm{PGL}_3^\delta(F)}$, $\chi_{\mathrm{PGL}_3^{\delta'}(F)}$	no
$\phi_{3.5.3d}$	$\chi\otimes\mathrm{St}_{\mathrm{PGL}_3}$, $\chi_{\mathrm{PGL}_3^\delta(F)}$, $\chi_{\mathrm{PGL}_3^{\delta'}(F)}$		yes

3.5.1. *Langlands correspondence.* Let T be the standard maximal torus in PGL_3 and let B_3 be the standard Borel subgroup of GL_3. We denote the matching positive roots of GL_3 by α_1 and α_2. Let P_{α_i} be the standard maximal parabolic subgroup of GL_3 such that the root space α_i is in the Levi of P_{α_i}. Since $\mathrm{PGL}_3(F) = \mathrm{GL}_3(F)/F^\times$, a representation of $\mathrm{PGL}_3(F)$ is just a representation of $\mathrm{GL}_3(F)$ with trivial central character. The cases treated below group representations together by infinitesimal parameter.

3.5.0 Let χ_1, χ_2 are characters of $F^\times$ such that $\chi_1\chi_2^{-1}$, $\chi_1^2\chi_2$ and $\chi_1\chi_2^2$ are not equal to ν and not equal to ν^{-1}. Then the induced representation

$$\mathrm{Ind}_{B_3(F)}^{\mathrm{GL}_3(F)}(\chi_1 \otimes \chi_2 \otimes \chi_1^{-1}\chi_2^{-1})$$

is irreducible. Note that this representation has trivial central character and thus can be viewed as a representation of $\mathrm{PGL}_3(F)$. We denote this representation of $\mathrm{PGL}_3(F)$ by $I^{\mathrm{PGL}_3}(\chi_1 \otimes \chi_2)$. The corresponding Langlands parameter for $I^{\mathrm{PGL}_3}(\chi_1 \otimes \chi_2)$ is given by

$$\phi_{3.5.0}(w, x) = \begin{pmatrix} \varphi_1(w) & 0 & 0 \\ 0 & \varphi_2(w) & 0 \\ 0 & 0 & \varphi_1(w)^{-1}\varphi_2(w)^{-1} \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C}),$$

where $\varphi_i : W_F \to \mathbb{C}^\times$ corresponds to $\chi_i : F^\times \to \mathbb{C}^\times$ under class field theory.

3.5.1 Let χ be a character of $F^\times$ with $\chi^3 \neq \nu^{\pm 1/2}, \nu^{\pm 3/2}$. We consider the representation

$$\mathrm{Ind}_{B_3(F)}^{\mathrm{GL}_3(F)}(\chi\nu^{1/2} \otimes \chi\nu^{-1/2} \otimes \chi^{-2})$$

of $\mathrm{GL}_3(F)$, which is also viewed as a representation of $\mathrm{PGL}_3(F)$. This representation has length 2, and its two irreducible components are $I_{\alpha_1}(\chi\mathrm{St}_{\mathrm{GL}_2} \otimes \chi^{-2})$ and $I_{\alpha_1}(\chi \circ \det \otimes \chi^{-2})$, which, adapting the notation above, are written as

$$I_{\alpha_1}^{\mathrm{PGL}_3}(\chi\mathrm{St}_{\mathrm{GL}_2}), \qquad \text{and} \qquad I_{\alpha_1}^{\mathrm{PGL}_3}(\chi \circ \det).$$

Here $I_{\alpha_1}^{\mathrm{PGL}_3}$ stands for $\mathrm{Ind}_{P_{\alpha_1}}^{\mathrm{GL}_3}$, and a representation of $\mathrm{GL}_3(F)$ is viewed as a representation of $\mathrm{PGL}_3(F)$ if it has trivial central character. The Langlands parameter of $I_{\alpha_1}(\chi \circ \det \otimes \chi^{-2})$ is given by

$$\phi_{3.5.1.a}(w, x) = \begin{pmatrix} \varphi(w)|w|^{1/2} & 0 & 0 \\ 0 & \varphi(w)|w|^{-1/2} & 0 \\ 0 & 0 & \varphi(w)^{-2} \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C}),$$

where $\varphi : W_F \to \mathbb{C}^\times$ is the dual character of χ. The Langlands parameter for $I_{\alpha_1}(\chi\mathrm{St}_{\mathrm{GL}_2} \otimes \chi^{-2})$ is given by

$$\phi_{3.5.1b}\left(w, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \begin{pmatrix} \varphi(w)a & \varphi(w)b & 0 \\ \varphi(w)c & \varphi(w)d & 0 \\ 0 & 0 & \varphi(w)^{-2} \end{pmatrix} = \begin{pmatrix} \varphi(w)x & \\ & \varphi(w)^{-2} \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C}).$$

Note that the infinitesimal parameter for these representations is

$$\lambda_{3.5.1a}(w) = \begin{pmatrix} \varphi(w)|w|^{1/2} & 0 & 0 \\ 0 & \varphi(w)|w|^{-1/2} & 0 \\ 0 & 0 & \varphi(w)^{-2} \end{pmatrix}.$$

3.5.2 Let χ be a character of $F^\times$ with $\chi^3 = 1$. Consider the representation

$$\mathrm{Ind}_{B_3(F)}^{\mathrm{GL}_3(F)}(\chi\nu^{2/3} \otimes \chi\nu^{-1/3} \otimes \chi\nu^{-1/3}),$$

which is also viewed as a representation of $\mathrm{PGL}_3(F)$. This representation has length 2 and its two components are

$$I_{\alpha_1}(\chi\nu^{1/6}\mathrm{St}_{\mathrm{GL}_2} \otimes \chi\nu^{-1/3}) \quad \text{and} \quad I_{\alpha_1}(\chi\nu^{1/6} \circ \det \otimes \chi\nu^{-1/3}).$$

The Langlands parameter of $I_{\alpha_1}(\chi\nu^{1/6}\circ\det\otimes\chi\nu^{-1/3})$ is given by

$$\phi_{3.5.2a}(w,x)=\begin{pmatrix}\varphi(w)|w|^{2/3} & 0 & 0\\ 0 & \varphi(w)|w|^{-1/3} & 0\\ 0 & 0 & \varphi(w)|w|^{-1/3}\end{pmatrix}\in \mathrm{SL}_3(\mathbb{C}),$$

and the Langlands parameter of $I_{\alpha_1}(\chi\nu^{1/6}\mathrm{St}_{\mathrm{GL}_2}\otimes\chi\nu^{-1/3})$ is given by

$$\phi_{3.5.2b}(w,x)=\begin{pmatrix}\varphi(w)|w|^{1/6}x & \\ & \varphi(w)|w|^{-1/3}\end{pmatrix}\in \mathrm{SL}_3(\mathbb{C}).$$

3.5.3 Let χ be a character of $F^\times$ with $\chi^3=1$. Consider the induced representation

$$\mathrm{Ind}_{B_3(F)}^{\mathrm{GL}_3(F)}(\chi\nu\otimes\chi\otimes\chi\nu^{-1})$$

of $\mathrm{GL}_3(F)$ (and also of $\mathrm{PGL}_3(F)$). This representation has length 4 and its components are given by

$$\begin{array}{cc}\chi_{\mathrm{PGL}_3(F)}, & \chi\otimes J_{\alpha_2}(\nu\otimes\nu^{-1/2}\mathrm{St}_{\mathrm{GL}_2}),\\ \chi\otimes J_{\alpha_1}(\nu^{1/2}\mathrm{St}_{\mathrm{GL}_2}\otimes\nu^{-1}), & \chi\otimes\mathrm{St}_{\mathrm{PGL}_3},\end{array}$$

where $\chi_{\mathrm{PGL}_3(F)}=\chi\circ\det$, $J_{\alpha_1}(\nu^{1/2}\mathrm{St}_{\mathrm{GL}_2}\otimes\nu^{-1})$ is the unique quotient of $I_{\alpha_1}(\nu^{1/2}\mathrm{St}_{\mathrm{GL}_2}\otimes\nu^{-1})$ and $J_{\alpha_2}(\nu\otimes\nu^{-1/2}\mathrm{St}_{\mathrm{GL}_2}$ is defined similarly. The local Langlands parameters of these 4 representations are given, in order, by

$$\begin{aligned}\phi_{3.5.3a}(w,x)&=\varphi(w)\begin{pmatrix}|w| & & \\ & 1 & \\ & & |w|^{-1}\end{pmatrix}\in \mathrm{SL}_3(\mathbb{C}),\\ \phi_{3.5.3b}(w,x)&=\varphi(w)\begin{pmatrix}|w| & \\ & |w|^{-1/2}x\end{pmatrix}\in \mathrm{SL}_3(\mathbb{C}),\\ \phi_{3.5.3c}(w,x)&=\varphi(w)\begin{pmatrix}|w|^{1/2}x & \\ & |w|^{-1}\end{pmatrix}\in \mathrm{SL}_3(\mathbb{C}),\\ \phi_{3.5.3d}(w,x)&=\varphi(w)\,\mathrm{Sym}^2(x)\in \mathrm{SL}_3(\mathbb{C}),\end{aligned}$$

where $\varphi: W_F\to C^\times$ is Langlands parameter for the quadratic character χ and Sym^2 is the symmetric square representation of $\mathrm{SL}_2(\mathbb{C})$.

This completes the description of the Langlands correspondence for unipotent representations of $\mathrm{PGL}_3(F)$.

3.5.2. *ABV-packets.* Table 3.5.1 presents the L-packets for all unipotent representations of $\mathrm{PGL}_3(F)$. Returning to pure L-packets, we only explain the case (3.5.3) since the others proceed by similar, simpler arguments. Denote the last Langlands parameter in Case (3.5.3) by $\phi_{(3.5.3)d}$. This Langlands parameter has component group μ_3 so its pure L-packet contains 3 admissible irreducible representations:

$$\Pi^{\mathrm{pure}}_{\phi_{3.5.3d}}(\mathrm{PGL}_3/F)=\{\chi\otimes\mathrm{St}_{\mathrm{PGL}_3},\chi_{\mathrm{PGL}_3^\delta(F)},\chi_{\mathrm{PGL}_3^{\delta'}(F)}\},$$

where $\chi_{\mathrm{PGL}_3^\delta(F)}$ (resp. $\chi_{\mathrm{PGL}_3^{\delta'}(F)}$) is the character of $\mathrm{PGL}_3^\delta(F)$ (resp. $\mathrm{PGL}_3^{\delta'}(F)$) obtained by composing χ with the reduced norm.

Table 3.5.2 presents the ABV-packet coefficients for the Langlands parameter $\phi_{(3.5.3)}$. From Table 3.5.2 we read off the ABV-packets in Case (3.5.3); the result appears in Table 3.5.1, where $\mathcal{D}$ (resp. $\mathcal{D}'$) is the character of $A_{\phi_{3.5.3d}}$ corresponding to the pure inner form δ (resp. δ') in $Z^1(F,\mathrm{SO}_4)$.

TABLE 3.5.2. ABV-packet coefficients $\langle\ ,\ \rangle$ for unipotent representations of PGL_3 and its pure inner forms with infinitesimal parameter $\lambda_{3.5.3}$. The first four rows refer to representations of $\mathrm{PGL}_3(F)$, while the fifth and sixth rows are refer to the pure inner forms δ and δ', respectively. Below, we identify δ (resp. δ') with the character of $A_{\phi_{3.5.3d}}$ corresponding to the pure inner form δ (resp. δ') in $Z^1(F, \mathrm{PGL}_3)$. All microlocal fundamental groups are order 3

Irreps with inf parameter $\lambda_{3.5.3}$	$\widehat{A^{\mathrm{ABV}}_{\phi_{3.5.3a}}}$	$\widehat{A^{\mathrm{ABV}}_{\phi_{3.5.3b}}}$	$\widehat{A^{\mathrm{ABV}}_{\phi_{3.5.3c}}}$	$\widehat{A^{\mathrm{ABV}}_{\phi_{3.5.3d}}}$
$\chi_{\mathrm{PGL}_3(F)}$	$\mathbb{1}$	0	0	0
$\chi\otimes J_{\alpha_1}(\nu\otimes\nu^{-1/2}\mathrm{St}_{\mathrm{GL}_2})$	0	$\mathbb{1}$	0	0
$\chi\otimes J_{\alpha_1}(\nu^{1/2}\mathrm{St}_{\mathrm{GL}_2}\otimes\nu^{-1})$	0	0	$\mathbb{1}$	0
$\chi\otimes\mathrm{St}_{\mathrm{PGL}_3}$	0	0	0	$\mathbb{1}$
$\chi_{\mathrm{PGL}_3^{\delta}(F)}$	δ	δ	δ	δ
$\chi_{\mathrm{PGL}_3^{\delta'}(F)}$	δ'	δ'	δ'	δ'

We now explain the calculations behind Table 3.5.2, as a special case of the techniques of [CFM+21]; see also [CFZ]. Let $\lambda_{3.5.3} : W_F \to \mathrm{SL}_3(\mathbb{C})$ be the infinitesimal parameter of $\phi_{3.5.3d}$:

$$\lambda_{3.5.3}(w) := \phi_{3.5.3d}\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2}\end{pmatrix}\right) = \begin{pmatrix} |w| & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |w|^{-1}\end{pmatrix}.$$

The moduli space of Langlands parameters with infinitesimal parameter $\lambda_{3.5.3}$ is the direct sum $\mathfrak{u}_{\hat\alpha_1}\oplus\mathfrak{u}_{\hat\alpha_2}$ of the root space for the simple roots $\hat\alpha_1$ and $\hat\alpha_2$, equipped with the group action of $\widehat{T}$ given by $t\cdot(x_1,x_2) := (\hat\alpha_1(t)x_1, \hat\alpha_2(t)x_2)$.

Vogan's version of the Langlands correspondence establishes a bijection between simple objects in the category $\mathsf{Per}_{\widehat{T}}(\mathfrak{u}_{\hat\alpha_1}\oplus\mathfrak{u}_{\hat\alpha_2})$ of $\widehat{T}$-equivariant perverse sheaves on $\mathfrak{u}_{\hat\alpha_1}\oplus\mathfrak{u}_{\hat\alpha_2}$ and irreducible admissible representations of PGL_3 and its pure inner forms with infinitesimal parameter $\lambda_{(3.5.3)}$. We have already seen there are six of the latter – the four irreducible subquotients of $\mathrm{Ind}_{B_3(F)}^{\mathrm{GL}_3(F)}(\chi\nu\otimes\chi\otimes\mu\nu^{-1})$ together with the characters $\chi_{\mathrm{PGL}_3^{\delta}(F)}$ and $\chi_{\mathrm{PGL}_3^{\delta'}(F)}$. Let us enumerate the corresponding simple objects in $\mathsf{Per}_{\widehat{T}}(\mathfrak{u}_{\hat\alpha_1}\oplus\mathfrak{u}_{\hat\alpha_2})$. Let $C_{3.5.3a}$ be the trivial orbit in $V_{\lambda_{3.5.3b}}$, let $C_{3.5.3b}$ be the $\widehat{T}$-orbit of $X_{\hat\alpha_1}$, which is the point in the moduli space for the Langlands parameter ϕ_1 for $\chi\otimes J_{\alpha_1}(\nu\otimes\nu^{-1/2}\mathrm{St}_{\mathrm{GL}_2})$; let $C_{3.5.3c}$ be the $\widehat{T}$-orbit of $X_{\hat\alpha_2}$, which is the point in the moduli space for the Langlands parameter ϕ_2 for $\chi\otimes J_{\alpha_1}(\nu^{1/2}\mathrm{St}_{\mathrm{GL}_2}\otimes\nu^{-1})$. Finally, let $C_{3.5.3d}$ be the $\widehat{T}$-orbit of $X_{\hat\alpha_1}+X_{\hat\alpha_2}$; this orbit of the Langlands parameter $\phi_{3.5.3d}$ for $\mu\otimes\mathrm{St}_{\mathrm{PGL}_3}$. The equivariant fundamental groups of these orbits are trivial with the exception of the open orbit $C_{3.5.3d}$, which has equivariant fundamental group

$$A_{\phi_{3.5.3d}} := \pi_1^{\widehat{T}}(C_{3.5.3}, X_{\hat\alpha_1}+X_{\hat\alpha_2}) = \pi_0(Z_{\widehat{T}}(X_{\hat\alpha_1}+X_{\hat\alpha_2})) = \{1,\theta_3,\theta_3^2\}.$$

The bijection between $\Pi^{\mathrm{pure}}_{\lambda_{3.5.3}}(\mathrm{PGL}_3/F)$ and the six simple perverse sheaves in $\mathsf{Per}_{\widehat{T}}(\mathfrak{u}_{\hat\alpha_1}\oplus\mathfrak{u}_{\hat\alpha_2})$ are given by the following table.

	$C_{3.5.3a}$	$C_{3.5.3b}$	$C_{3.5.3c}$	$C_{3.5.3d}$
$\chi_{\mathrm{PGL}_3(F)}$	$\mathcal{IC}(\mathbb{1}_{C_{3.5.3a}})$			
$\chi\otimes J_{\alpha_1}(\nu\otimes\nu^{-1/2}\mathrm{St}_{\mathrm{GL}_2})$		$\mathcal{IC}(\mathbb{1}_{C_{3.5.3b}})$		
$\chi\otimes J_{\alpha_1}(\nu^{1/2}\mathrm{St}_{\mathrm{GL}_2}\otimes\nu^{-1})$			$\mathcal{IC}(\mathbb{1}_{C_{3.5.3c}})$	
$\chi\otimes\mathrm{St}_{\mathrm{PGL}_3}$				$\mathcal{IC}(\mathbb{1}_{C_{3.5.3d}})$
$\chi_{\mathrm{PGL}_3^{\delta}(F)}$				$\mathcal{IC}(\mathcal{D}_{C_{3.5.3d}})$
$\chi_{\mathrm{PGL}_3^{\delta'}(F)}$				$\mathcal{IC}(\mathcal{D}'_{C_{3.5.3d}})$,

where $\mathcal{D}$ (resp. $\mathcal{D}'$) is the local system corresponding to the choice of primitive cube root of unity δ (resp. δ') appearing in the pure inner form $\delta \in Z^1(F, \mathrm{PGL}_3)$.

Now, for each irreducible admissible π above, and for each Langlands parameter ϕ with infinitesimal parameter $\lambda_{(3.5.3)}$, we compute $\mathsf{NEvs}_{C_\phi} \mathcal{P}(\pi)$, where $\mathcal{P}(\pi)$ is the corresponding simple object in $\mathsf{Per}_{\widehat{T}}(\mathfrak{u}_{\hat\alpha_1} \oplus \mathfrak{u}_{\hat\alpha_2})$. The result is a local system in $\Lambda^{\mathrm{sreg}}_{C_\phi} := T^*_{C_\phi}(\mathfrak{u}_{\hat\alpha_1} \oplus \mathfrak{u}_{\hat\alpha_2})_{\mathrm{sreg}}$ and this may be viewed as a representation of the equivariant fundamental group of $\Lambda^{\mathrm{sreg}}_{C_\phi}$, which is A^{ABV}_ϕ. The results are summarized in Table 3.5.2.

The properties in Theorems 2.2 and 2.14 are elementary for $\mathrm{PGL}_3(F)$ and its pure inner forms as is [CFM+21, Conjecture 1].

4. Geometric endoscopy

4.1. **Lifting Langlands parameters.** Let (G, s, ξ) be an endoscopic triple for G_2 and let $\phi : W'_F \to {}^LG$ be an unramified Langlands parameter such that $s \in \mathcal{S}^{\mathrm{ABV}}_{\xi\circ\phi}$. The Langlands parameter ϕ determines, or lifts to, a Langlands parameter $\xi \circ \phi : W'_F \to {}^LG_2$. Matching the cases that appear in our classification of Langlands parameters for G and G_2 is a little complicated, because we have grouped parameters into families based on the geometry created by the corresponding infinitesimal parameters. Consequently, it happens that different members of these families may lift to different families for G_2. To illustrate this phenomenon, here we give the details for one of the six endoscopic groups for G_2. Together with all other cases, this information is summarized in Table 4.1.2.

(A$_2$) $G = \mathrm{PGL}_3$, $s = \hat{m}(\theta_3^2, \theta_3)$.

(0) The Langlands parameter $\phi_{3.5.0}$ lifts to ϕ_0 iff $\varphi_1(\mathrm{Fr}), \varphi_2(\mathrm{Fr}), \varphi_1(\mathrm{Fr})\varphi_2(\mathrm{Fr}) \notin \{q^{\pm 1}\}$; lifts to ϕ_{1a} iff one of $\varphi_1(\mathrm{Fr}), \varphi_2(\mathrm{Fr}), \varphi_1(\mathrm{Fr})\varphi_2(\mathrm{Fr})$ is in $\{q^{\pm 1}\}$; and lifts to ϕ_{5a} iff two of $\varphi_1(\mathrm{Fr}), \varphi_2(\mathrm{Fr}), \varphi_1(\mathrm{Fr})\varphi_2(\mathrm{Fr})$ are in $\{q^{\pm 1}\}$.

(1) The Langlands parameter $\phi_{3.5.1a}$ (resp. $\phi_{3.5.1a}$) lifts to ϕ_{2a} (resp. ϕ_{2b}) iff $\varphi(\mathrm{Fr}) \notin \{-q^{\pm 1/2}, q^{\pm 3/2}\}$; ϕ_{4a} (resp. ϕ_{4b}) iff $\varphi(\mathrm{Fr}) \in \{-q^{\pm 1/2}\}$; ϕ_{7a} (resp. ϕ_{7b}) iff $\varphi(\mathrm{Fr}) \in \{q^{\pm 3/2}\}$.

(2) The Langlands parameter $\phi_{3.5.2a}$ (resp. $\phi_{3.5.2b}$) lifts to ϕ_{3a} (resp. ϕ_{3b}).

(3) The Langlands parameter $\phi_{3.5.3a}$ (resp. $\phi_{3.5.3b}$, $\phi_{3.5.3c}$, $\phi_{3.5.3d}$) lifts to ϕ_{6a} (resp. ϕ_{6b}, ϕ_{6b}, ϕ_{6b}) iff $\varphi(\mathrm{Fr}) \neq 1$; and lifts to ϕ_{8a} (resp. ϕ_{8b}, ϕ_{8c}, ϕ_{8d}) iff $\varphi(\mathrm{Fr}) = 1$.

If we turn the problem of lifting parameters on its head, our classification of scheme for Langlands parameters is more illuminating: instead of lifting parameters from endoscopic groups, begin with a Langlands parameter for G_2 and find all endoscopic triples (G, s, ξ) such that the Langlands parameter factors through $\xi : {}^LG \to {}^LG_2$. This information appears in Table 4.1.1.

4.2. **Regular conormal vectors and Arthur parameters.** Let (G, s, ξ) be an endoscopic triple for G_2 and let $\phi : W'_F \to {}^LG$ be an unramified Langlands parameter. Let λ be the infinitesimal parameter for ϕ; then $\xi \circ \lambda$ is the infinitesimal parameter for $\xi \circ \phi$. Now consider the map $\xi : {}^LG \to {}^LG_2$ on dual groups and the induced homomorphism $\widehat{G} \to \widehat{G_2}$, also denoted by ξ below. Pass to tangent spaces $d\xi : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}_2}$ and restricted to Vogan varieties to define the embedding

$$d\xi|_{V_\lambda} : V_\lambda \to V_{\xi\circ\lambda}$$

such that the subvariety V_λ is the fixed-point set of the $(V_{\xi\circ\lambda})^s = V_\lambda$. This induces

$$d\xi|_{T^*(V_\lambda)} : T^*(V_\lambda) \to T^*(V_{\xi\circ\lambda})$$

such that $(T^*(V_{\xi\circ\lambda}))^s = T^*(V_\lambda)$; recal that we may view $T^*(V_\lambda)$ as a subvariety of $\widehat{\mathfrak{g}}$. Likewise, this induces a map of conormal bundles

$$d\xi|_{\Lambda_\lambda} : \Lambda_\lambda \to \Lambda_{\xi\circ\lambda}$$

TABLE 4.1.1. Langlands parameters and their associated minimal endoscopic groups. In the column labeled "minimal endoscopic group(s)" we indicate, for each Langlands parameter ϕ of G_2, the minimal endoscopic groups G such that the parameter factors ϕ through $\xi : {}^LG \to {}^LG_2$. The column labeled "Arthur type" gives necessary and sufficient conditions for ϕ to be of Arthur type.

Langlands Parameter ϕ	Minimal Endoscopic Group(s)	Arthur Type Parameter	Component groups A_ϕ	A_ϕ^{ABV}	$\dim(\phi)$
ϕ_0	T	$a_1 = a_2 = 0$	1	1	0
ϕ_{1a}	T	$a = 1/2$	1	1	0
ϕ_{1b}	$\mathrm{GL}_2^{\gamma_2}$	$a = 1/2$	1	1	1
ϕ_{2a}	T	$a = 1/2$	1	1	0
ϕ_{2b}	$\mathrm{GL}_2^{\gamma_1}$	$a = 1/2$	1	1	1
$\phi_{3.a}$	T	no	1	1	0
$\phi_{3.b}$	$\mathrm{GL}_2^{\gamma_1}$	no	1	1	2
ϕ_{4a}	T	yes	1	$\langle\theta_2\rangle$	0
ϕ_{4b}	$\mathrm{GL}_2^{\gamma_1}$	yes	1	$\langle\theta_2\rangle$	1
ϕ_{4c}	$\mathrm{GL}_2^{\gamma_2}$	yes	1	$\langle\theta_2\rangle$	1
ϕ_{4d}	SO_4	yes	$\langle\theta_2\rangle$	$\langle\theta_2\rangle$	2
ϕ_{5a}	T	no	1	1	0
ϕ_{5b}	$\mathrm{GL}_2^{\gamma_2}$	no	1	1	2
ϕ_{6a}	T	yes	1	$\langle\theta_3\rangle$	0
ϕ_{6b}	$\mathrm{GL}_2^{\gamma_1}$	no	1	$\langle\theta_3\rangle$	1
ϕ_{6c}	$\mathrm{GL}_2^{\gamma_1}$	no	1	$\langle\theta_3\rangle$	1
ϕ_{6d}	PGL_3	yes	$\langle\theta_3\rangle$	$\langle\theta_3\rangle$	2
ϕ_{7a}	T	yes	1	1	0
ϕ_{7b}	$\mathrm{GL}_2^{\gamma_1}$	no	1	1	1
ϕ_{7c}	$\mathrm{GL}_2^{\gamma_2}$	no	1	1	1
ϕ_{7d}	G_2	yes	1	1	2
ϕ_{8a}	T	yes	1	S_3	0
ϕ_{8b}	$\mathrm{GL}_2^{\gamma_1}$	yes	1	$\langle\theta_2\rangle$	2
ϕ_{8c}	$\mathrm{GL}_2^{\gamma_2}$	yes	1	$\langle\theta_2\rangle$	3
ϕ_{8d}	$\mathrm{SO}_4, \mathrm{PGL}_3$	yes	S_3	S_3	4

such that $\Lambda_\lambda = (\Lambda_{\xi\circ\lambda})^s$; again, we may view Λ_λ as a subvariety of $\widehat{\mathfrak{g}}$. Now let x_ϕ (resp. $x_{\xi\circ\phi}$) be the point in the moduli space V_λ (resp. $V_{\xi\circ\lambda}$) for $\phi : W'_F \to {}^LG$ (resp. for $\xi\circ\phi : W'_F \to {}^LG_2$) and let C_ϕ (resp. $C_{\xi\circ\phi}$) be its H_λ-orbit (resp. $H_{\xi\circ\lambda}$-orbit). Now $d\xi|_{\Lambda_\lambda}$ restricts to an immersion

$$d\xi|_{\Lambda_{\lambda,C_\phi}} : \Lambda_{\lambda,C_\phi} \to \Lambda_{\xi\circ\lambda,C_{\xi\circ\phi}}. \tag{13}$$

Now consider the restriction

$$\begin{array}{ccc} \Lambda_{\lambda,C_\phi} & \xrightarrow{d\xi|_{\Lambda_{\lambda,C_\phi}}} & \Lambda_{\xi\circ\lambda,C_{\xi\circ\phi}} \\ \uparrow & \nearrow_{d\xi|_{\Lambda^{\text{sreg}}_{\lambda,C_\phi}}} & \\ \Lambda^{\text{sreg}}_{\lambda,C_\phi} & & \end{array}$$

TABLE 4.1.2. Summary of possible lifts for members of each family of Langlands parameters. The leftmost column gives endoscopic groups G for G_2. The column "Endoscopic Langlands parameters" refers to Sections 3.2 through 3.4. The column "Arthur-type" indicates those ξ-conormal parameters that, when unitary, are parameters of Arthur type; see Definition 6 for this notion. The lifts appearing in the column "other regular" indicates all other ξ-conormal parameters. In these last two cases, the information in these two columns refers to the classification of unramified Langlands parameters for G_2 appearing in Section 1.2. Finally, the column "irregular lifts" indicates all remaining lifts.

		s-regular lifts		
Endoscopic group	Endoscopic Langlands parameters	Arthur type	Other regular	Irregular lifts
(T)				
	$\phi_{3.2.0}$	ϕ_0		$\phi_{1a}, \phi_{2a}, \phi_{3a}, \phi_{4a}, \phi_{5a}, \phi_{6a}, \phi_{7a}, \phi_{8a}$
(A_1^{s})				
	$\phi_{3.3.0}$	ϕ_0		$\phi_{1a}, \phi_{2a}, \phi_{3a}, \phi_{4a}, \phi_{5a}, \phi_{6a}, \phi_{7a}, \phi_{8a}$
	$\phi_{3.3.1a}$	ϕ_{1a}	ϕ_{5a}	$\phi_{4a}, \phi_{7a}, \phi_{8a}$
	$\phi_{3.3.1b}$	ϕ_{1b}	ϕ_{5b}	$\phi_{4c}, \phi_{7c}, \phi_{8c}$
(A_1^{l})				
	$\phi_{3.3.0}$	ϕ_0		$\phi_{1a}, \phi_{2a}, \phi_{3a}, \phi_{4a}, \phi_{5a}, \phi_{6a}, \phi_{7a}, \phi_{8a}$
	$\phi_{3.3.1a}$	ϕ_{2a}	ϕ_{3a}	$\phi_{4a}, \phi_{6a}, \phi_{7a}, \phi_{8a}$
	$\phi_{3.3.1b}$	ϕ_{2b}	ϕ_{3b}	$\phi_{4b}, \phi_{6b}, \phi_{7b}, \phi_{8b}$
(D_2)				
	$\phi_{3.4.0}$	ϕ_0		$\phi_{1a}, \phi_{2a}, \phi_{3a}, \phi_{4a}, \phi_{5a}, \phi_{6a}, \phi_{7a}, \phi_{8a}$
	$\phi_{3.4.1a}$	ϕ_{1a}	ϕ_{5a}	ϕ_{7a}
	$\phi_{3.4.1b}$	ϕ_{1b}	ϕ_{5b}	ϕ_{7c}
	$\phi_{3.4.2a}$	ϕ_{2a}	ϕ_{3a}	ϕ_{6a}, ϕ_{7a}
	$\phi_{3.4.2b}$	ϕ_{2b}	ϕ_{3b}	ϕ_{6b}, ϕ_{7b}
	$\phi_{3.4.3a}$	ϕ_{4a}, ϕ_{8a}		
	$\phi_{3.4.3b}$	ϕ_{4b}, ϕ_{8b}		
	$\phi_{3.4.3c}$	ϕ_{4c}, ϕ_{8c}		
	$\phi_{3.4.3d}$	ϕ_{4d}, ϕ_{8d}		
(A_2)				
	$\phi_{3.5.0}$	ϕ_0		ϕ_{1a}, ϕ_{5a}
	$\phi_{3.5.1a}$	ϕ_{2a}		ϕ_{4a}, ϕ_{7a}
	$\phi_{3.5.1b}$	ϕ_{2b}		ϕ_{4b}, ϕ_{7b}
	$\phi_{3.5.2a}$		ϕ_{3a}	
	$\phi_{3.5.2b}$		ϕ_{3b}	
	$\phi_{3.5.3a}$	ϕ_{6a}, ϕ_{8a}		
	$\phi_{3.5.3b}$		ϕ_{6b}	ϕ_{8b}
	$\phi_{3.5.3c}$		ϕ_{6c}	ϕ_{8b}
	$\phi_{3.5.3d}$	ϕ_{6d}, ϕ_{8d}		

Definition 6. Let (G, s, ξ) be an endoscopic triple for G_2 and let $\phi : W'_F \to {}^LG$ be an unramified Langlands parameter. Let us say that a Langlands parameter $\phi : W'_F \to {}^LG$ is ξ-conormal, if the image of $d\xi|_{\Lambda^{\text{sreg}}_{\lambda,C_\phi}}$ is contained in $\Lambda^{\text{sreg}}_{\xi\circ\lambda,C_{\xi\circ\phi}}$. If ϕ is ξ-conormal the map

$$d\xi|_{\Lambda^{\text{sreg}}_{\lambda,C_\phi}} : \Lambda^{\text{sreg}}_{\lambda,C_\phi} \to \Lambda^{\text{sreg}}_{\xi\circ\lambda,C_{\xi\circ\phi}}$$

induces a group homomorphism of equivariant fundamental groups, henceforth denoted by

$$d\xi_\phi : A^{\text{ABV}}_\phi \to A^{\text{ABV}}_{\xi\circ\phi}.$$

Proposition 4.1. *Let (G, s, ξ) be an endoscopic triple for G_2 and let $\phi : W'_F \to {}^LG$ be an unramified Langlands parameter. If ϕ is of Arthur type then ϕ is ξ-conormal.*

Proof. Let $\psi : W''_F \to {}^LG$ be the Arthur parameter such that $\phi(w, x) = \psi(w, x, d_w)$. Set $x := d\psi(1, e, 1)$ and $y := d\psi(1, 1, f)$. Then by [CFM+21, Prop. 6.1.1] we have (x, y) is a strongly regular conormal vector for ϕ. Consider $\xi \circ \psi : W''_F \to {}^LG_2$. Then again by [CFM+21, Prop. 6.1.1] we have $d\xi(x, y) = (d(\xi \circ \psi)(1, e, 1), d(\xi \circ \psi)(1, 1, f))$ is a strongly regular conormal vector for $\xi \circ \phi$. □

In Table 4.1.2 we present a complete list of all endoscopic triples (G, s, ξ) and all unramified Langlands parameters $\phi : W'_F \to {}^LG_2$ for which ϕ is ξ-conormal. We now explain this table. The left-hand column of Table 4.1.2 lists the endoscopic groups G for G_2 as explained in Section 3.1. Then, for each such G, we list the families of endoscopic Langlands parameters $\phi : W'_F \to {}^LG$ as they appeared in Sections 3.2 through 3.4. In Section 4.1 we saw that for each ϕ in these families, there are multiple Langlands parameters for $G_2(F)$ to which ϕ may lift, depending on the properties of ϕ. Each row of Table 4.1.2 lists the lifts $\xi\circ\phi$ that arise as ϕ ranges through each family, partially ordered left to right along rows by the relative dimension $\dim(\xi \circ \phi) - \dim(\phi)$; note that if this relative dimension is 0 then ϕ is trivially ξ-conormal.

4.3. Statement of the Trace/fixed-point formula.

Theorem 4.2. *Let (G, s, ξ) be an endoscopic triple for G_2 and let $\phi : W'_F \to {}^LG$ be an unramified Langlands parameter for G. If ϕ is ξ-conormal then*

$$\operatorname{trace}_s \left(\mathsf{NEvs}_{\xi\circ\phi}[\dim(\xi \circ \phi)]\mathcal{P}\right) = \operatorname{trace}_s \left(\mathsf{NEvs}_\phi[\dim(\phi)] \operatorname{res}\mathcal{P}\right)$$

for every $\mathcal{P} \in \mathsf{Per}_{H_{\xi\circ\lambda}}(V_{\xi\circ\lambda})$.

The proof of Theorem 4.2 will occupy the rest of this Section, so we explain the strategy here. We work through the list of endoscopic triples (G, s, ξ) for G_2 and, in each case, we recall that classification of Langlands parameters $\phi : W'_F \to {}^LG$ in terms of infinitesimal parameters in Sections 3.2 through 3.4. We identify those Langlands parameters ϕ that are ξ-conormal, expanding on the information presented in Table 4.1.2. We then recall the classification of Vogan varieties $H_{\xi\circ\lambda} \times V_{\xi\circ\lambda} \to V_{\xi\circ\lambda}$ in terms of prehomogeneous vector spaces $H \times V \to V$ from Section 1.2 and we observe that same list of five prehomogeneous vector spaces capture all instances of the Vogan varieties $H_\lambda \times V_\lambda \to V_\lambda$. We further show that the embeddings

$$d\xi|_{V_\lambda} : V_\lambda \to V_{\xi\circ\lambda},$$

explained at the beginning of Section 4.2, all take the form

$$V^s \subseteq V$$

where V is one of these five prehomogeneous vector spaces and $s \in H$ has finite order (hence semisimple). We are then able to match L-parameters ϕ with H^s-orbits $C' \subseteq V^s$ and identify those that are ξ-conormal with the corresponding property of the H^s-orbit C'; we refer to these orbits as V-conormal. This is the genesis of the notion of ξ-conormal, in fact. We then prove the theorem for these prehomogeneous vector spaces in the form of Proposition 4.4, for all V-conormal orbits $C' \subseteq V^s$.

4.4. **Prehomogeneous vector subspaces.**

Lemma 4.3. *Let $H \times V \to V$ be one of the five prehomogenous vector spaces appearing in Proposition 1.2; observe that H is reductive in these cases. For semisimple $s \in H$, set $H^s := Z_H(s)$ and $V^s := \{x \in V \mid s(x) = x\}$. Then $H^s \times V^s \to V^s$ be is also a prehomogeneous vector space and again one of the five prehomogenous vector spaces appearing in Proposition 1.2.*

Proof. When $V = V^s$ and $H = H^s$ this is a tautology, whereas $V = V^s$ and $H \neq H^s$ only occurs for P0.

Up to the natural notion of equivalence, the possibilities for proper fix-point prehomogenous vector subspaces are the following.

P1 The prehomogeneous vector space $V = \mathbb{A}^1$ with $H = \mathrm{GL}_1$-action (scalar multiplication) has only one proper fixed-point prehomogeneous vector subspace:
  (i) $V^s = \{0\}$ with $H^s = \mathrm{GL}_1$ (this is P0), with, $s = -1$.

P2 The prehomogeneous vector space $V = \mathbb{A}^2$ with $H = \mathrm{GL}_2$-action given by twisted matrix multiplication $h.x = \det(h)^n\, hx$ also has the following proper fixed-point prehomogeneous vector subspaces:
  (i) $V^s = \{0\}$ with $H^s = \mathrm{GL}_2$ or GL_1^2 (this is P0), wolog $s = \mathrm{diag}(s_1, s_2)$ where s_1 and s_2 are primitive n-th roots of unity.;
  (ii) $V^s = \{(x_1, 0) \mid x_1\}$ and $H^s = \mathrm{GL}_1^2$ (this is equivalent to P1), wolog $s = \mathrm{diag}(1, s_2)$ where s_2 is a primitive n-th root of unity.

P3 The prehomogeneous vector space $V = \mathbb{A}^2$ with $H = \mathrm{GL}_1^2$-action given by $(t_1, t_2).(x_1, x_2) = (t_1 x_1, t_1 t_2^n x_2)$, for positive integer n, the following proper fixed-point prehomogeneous vector subspaces are possible:
  (i) $V^s = \{(0, 0)\}$ and $H^s = H$ (this is P0) for $s = (-1, s_2)$ where s_2 is not an n-th root of unity;
  (ii) $V^s = \{(x_1, 0) \mid x_1\}$ and $H^s = H$ (this is P1) for$s = (1, s_2)$ where s_2 is not an n-th root of unity;
  (iii) $V^s = \{(0, x_2) \mid x_2\}$ and $H^s = H$ (this is P1) $s = (-1, s_2)$ where s_2 is an n-th root of unity.

P4 The prehomogeneous vector space $V = \mathbb{A}^4$ with $H = \mathrm{GL}_2$ and action $\det^{-1} \otimes \mathrm{Sym}^3$ has the following proper fixed-point prehomogeneous vector subspaces, up to isomorphism:
  (i) $V^s = \{(0, 0, 0, 0)\}$ and $H^s = \mathrm{GL}_2$ or GL_1^2 (this is P0), wolog $s = \mathrm{diag}(-1, -1)$;
  (ii) $V^s = \{(x_1, 0, 0, 0) \mid x_1\}$ and $H^s = \mathrm{GL}_1^2$ (this is equivalent to P1), wolog $s = \mathrm{diag}(-1, i)$;
  (iii) $V^s = \{(0, x_2, 0, 0) \mid x_2\}$ and $H^s = \mathrm{GL}_1^2$ (this is equivalent to P1) for $s = \mathrm{diag}(1, i)$;
  (iv) $V^s = \{(x_1, 0, x_3, 0) \mid x_1, x_3\}$ and $H^s = \mathrm{GL}_1^2$ (this is P3 for $n = 2$) and $s = \mathrm{diag}(-1, 1)$;
  (v) $V^s = \{(x_1, 0, 0, x_4) \mid x_1, x_4\}$ and $H^s = \mathrm{GL}_1^2$ (this is P3 for $n = 3$) and $s = \mathrm{diag}(\theta_3^2, \theta_3)$.

□

4.5. V**-conormal.** Let $H \times V \to V$ be one of the prehomogeneous vector spaces appearing in Proposition 1.2 and let $H^s \times V^s \to V^s$ be one of the subspaces appearing in Lemma 4.3, for semisimple $s \in H$. Then $T^*(V)^s = T^*(V^s)$, where H acts on $T^*(V)$ by $s \cdot (x, y) := (s \cdot x, s \cdot y)$. Recall that $\Lambda = \{(x, y) \in T^*(V) \mid [x, y] = 0\}$; then $\Lambda^s = \{(x', y') \in T^*(V^s) \mid [x', y'] = 0\}$.

Now let $C' \subseteq V^s$ be an H^s-orbit. The H-orbit of C', denoted by $H \cdot C' \subseteq V$ and sometimes called the saturation of C', is a single H-orbit, denoted below by C. The intersection C^s of C with V^s contains C' but may contain other H^s-orbits: we write $C^s = \cup_i C'_i$. Then, $(\Lambda_C)^s = \cup_i (\Lambda^s)_{C'_i}$ Passing to the regular part of the conormal varieties we find that it can happen that $(\Lambda_C^{\mathrm{sreg}})^s$ and $\cup_i(\Lambda^s_{C'_i})^{\mathrm{sreg}}$ are not equal. Of course, this is essentially the same phenomenon discussed in Section 4.2. Definition 6 now takes this form.

Definition 7. With notation as above, let us say that an H^s-orbit $C' \subseteq V^s$ is V-conormal if $(\Lambda^s_{C'})^{\mathrm{sreg}} \subseteq (\Lambda_C^{\mathrm{sreg}})^s$, where the H-orbit $C \subseteq V$ is the saturation of C'. In this case, this inclusion induces a map of equivariant fundamental groups $A^{\mathrm{ABV}}_{C'} \to A^{\mathrm{ABV}}_C$.

Whenever $V = V^s$ and $H = H^s$ then we trivially have that every orbit is V-conormal. For those cases under consideration in this paper where $V = V^s$ and $H \neq H^s$ the same remains true. We note that this only occurs for P0.

From the cases of prehomogenous vector spaces V and proper subspaces V^s appearing in Lemma 4.3, only the following H^s-orbits $C' \subset V^s$ are V-conormal:

(1) Case P2(ii). The closed orbit C'_0 and the open orbit C'_1 in V^s are V-conormal. We note that the saturation of the closed H^s-orbit C'_0 in V^s is the closed H-orbit C_0 in V; the saturation of the open H^s-orbit C'_1 in V^s is the H-orbit C_1 in V, which is not open.
(2) Case P4(iv). Every H^s-orbit C' in V^s is V-conormal. These orbits are denoted by C'_0, C'_1, C'_2 and C'_3, where C'_0 is closed and C'_3 is open and where C'_1 is the H^s-orbit of $(1,0,0,0)$ and C'_2 is the H^s-orbit of $(0,0,1,0)$. The saturation of the closed H^s-orbit C'_0 is the closed H-orbit C_0, the saturation of C'_1 is C_1, the saturation of C'_2 is C_2 and the saturation of the open H^s-orbit C'_3 is the open H-orbit C_3.
(3) Case P4(v). Only two of the four H^s-orbits C' in V^s are V-conormal. Let the H^s-orbits in V^s be denoted by C'_0, C'_1, C'_2 and C'_3, where C'_0 is closed and C'_3 is open and where C'_1 is the H^s-orbit of $(1,0,0,0)$ while C'_2 is the H^s-orbit of $(0,0,0,1)$. Then the saturation of the closed H^s-orbit C'_0 is the closed H-orbit C_0, the saturation of C'_1 is C_1, the saturation of C'_2 is C_1 again, and the saturation of the open H^s-orbit C'_3 is the open H-orbit C_3. Only C'_0 and C'_3 are V-conormal.

4.6. **Restriction.** Let (G,s,ξ) be an endscopic triple for G_2 over F. Let λ be an infinitesimal parameter for G. To prove Theorem 4.2 we will calculate the equivariant restriction functor

$$\mathrm{res} : D_{H_\lambda}(V_{\xi\circ\lambda}) \to D_{H_\lambda}(V_\lambda)$$

on simple objects in the abelian category $\mathsf{Per}_H(V_{\xi\circ\lambda})$. As we will see, the restriction $\mathrm{res}\mathcal{P}$ of an equivariant perverse sheaf is not necessarily perverse. To address this issue, we replace $\mathsf{Per}_H(V_\lambda)$ with the full subcategory $\mathsf{Per}^\bullet_H(V_\lambda)$ of $D_H(V_\lambda)$ generated by shifts $\mathcal{P}[n]$ of equivariant perverse sheaves. In this section we will see that res defines

$$\mathrm{res} : \mathsf{Per}^\bullet_{H_{\xi\circ\lambda}}(V_{\xi\circ\lambda}) \to \mathsf{Per}^\bullet_{H_\lambda}(V_\lambda)$$

We use this fact to replace restriction along $d\xi : V_\lambda \to V_{\xi\circ\lambda}$ with restriction along $V^s \hookrightarrow V$, where $H \times V \to V$ is one of the five prehomogeneous vector spaces from Proposition 1.2 and $s \in H$ is semisimple. The functor

$$\mathrm{res} : \mathsf{Per}^\bullet_H(V) \to \mathsf{Per}^\bullet_{H^s}(V^s)$$

on simple objects is given in Table 4.6.1 for the three prehomogeneous vector spaces appearing semisimple $s \in H$ appearing in Section 4.5.

In Table 4.6.1 we find four indecomposable perverse sheaves, given here.

- On P3 (for $n = 2$) we define $\mathcal{F}_2 = \mathbb{1}_{C_2\cup C_1\cup C_0}[1]$ which is cohomologically equivalent to a complex

$$\cdots \to 0 \to (\mathbb{1}_{\overline{C_2}} \oplus \mathbb{1}_{\overline{C_1}}) \to \mathbb{1}_{C_0}$$

  and hence sits in the exact sequence

$$0 \to \mathcal{IC}(\mathbb{1}_{C_0}) \to \mathcal{F}_2 \to \mathcal{IC}(\mathbb{1}_{C_1}) \oplus \mathcal{IC}(\mathbb{1}_{C_2}) \to 0,$$

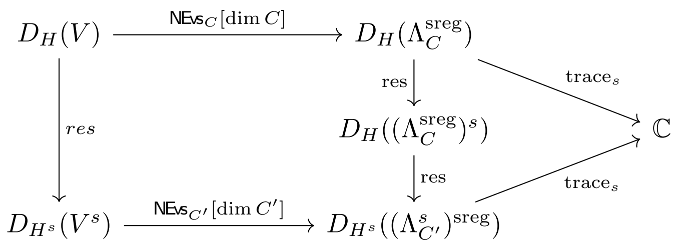
  which we use to compute the functor Ev on these perverse sheaves in Table 4.7.2.

TABLE 4.6.1. Restrictions of Standard and Simple Perverse Sheaves

Case	Standard Sheaf	Restriction	Perverse Sheaf	Restriction
P2(ii)	$\mathbb{1}_{C_0}$	$\mathbb{1}_{C_0'}$	$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathcal{IC}(\mathbb{1}_{C_0'})$
	$\mathbb{1}_{C_1}$	$\mathbb{1}_{C_1'}$	$\mathcal{IC}(\mathbb{1}_{C_1})$	$\mathcal{IC}(\mathbb{1}_{C_1'})[1]$
P4(iv)	$\mathbb{1}_{C_0}$	$\mathbb{1}_{C_0'}$	$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathcal{IC}(\mathbb{1}_{C_0'})$
	$\mathbb{1}_{C_1}$	$\mathbb{1}_{C_1'}$	$\mathcal{IC}(\mathbb{1}_{C_1})$	$\mathcal{IC}(\mathbb{1}_{C_1'})[1]$
	$\mathbb{1}_{C_2}$	$\mathbb{1}_{C_2'}$	$\mathcal{IC}(\mathbb{1}_{C_2})$	$\mathcal{F}_2[2] \oplus \mathcal{IC}(\mathbb{1}_{C_0'})[1]$
	$\mathbb{1}_{C_3}$	$\mathbb{1}_{C_3'}$	$\mathcal{IC}(\mathbb{1}_{C_3})$	$\mathcal{IC}(\mathbb{1}_{C_3'})[2]$
	ϱ_{C_3}	$\mathbb{1}_{C_3'} \oplus \rho_{C_3'}$	$\mathcal{IC}(\varrho_{C_3})$	$\mathcal{F}_3[2] \oplus \mathcal{IC}(\mathbb{1}_{C_0'})[2] \oplus \mathcal{IC}(\rho_{C_3'})[2]$
	ε_{C_3}	$\rho_{C_3'}$	$\mathcal{IC}(\varepsilon_{C_3})$	$\mathcal{IC}(\rho_{C_3'})[2]$
P4(v)	$\mathbb{1}_{C_0}$	$\mathbb{1}_{C_0'}$	$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathcal{IC}(\mathbb{1}_{C_0'})$
	$\mathbb{1}_{C_1}$	$\mathbb{1}_{C_1'} \oplus \mathbb{1}_{C_2'}$	$\mathcal{IC}(\mathbb{1}_{C_1})$	$\mathcal{F}_4[1]$
	$\mathbb{1}_{C_2}$	0	$\mathcal{IC}(\mathbb{1}_{C_2})$	$\mathcal{F}_4[2] \oplus \mathcal{IC}(\mathbb{1}_{C_0'})[1]$
	$\mathbb{1}_{C_3}$	$\mathbb{1}_{C_3'}$	$\mathcal{IC}(\mathbb{1}_{C_3})$	$\mathcal{IC}(\mathbb{1}_{C_3'})[2]$
	ϱ_{C_3}	$\rho_{C_3'} \oplus \rho^2_{C_3'}$	$\mathcal{IC}(\varrho_{C_3})$	$\mathcal{IC}(\rho_{C_3'})[2] \oplus \mathcal{IC}(\rho^2_{C_3'})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_0'})[2]$
	ε_{C_3}	$\mathbb{1}_{C_3'}$	$\mathcal{IC}(\varepsilon_{C_3})$	$\mathcal{F}_5[2]$

- On P3 (for $n = 2$) we define $\mathcal{F}_3 = \mathbb{1}_{C_3 \cup C_2}[2]$ which is cohomologically equivalent to a complex
$$\cdots \to 0 \to \mathbb{1}_{\overline{C_3}} \to 1_{\overline{C_1}} \to 0$$
and hence sits in the exact sequence
$$0 \to \mathcal{IC}(\mathbb{1}_{C_1}) \to \mathcal{F}_3 \to \mathcal{IC}(\mathbb{1}_{C_3}) \to 0,$$
which we use to compute the functor Ev on these perverse sheaves in Table 4.7.2.
- On P3 (for $n = 2$) we define $\mathcal{F}_4 = \mathbb{1}_{C_2 \cup C_1 \cup C_0}[1]$ which is cohomologically equivalent to a complex
$$\cdots \to 0 \to (\mathbb{1}_{\overline{C_2}} \oplus \mathbb{1}_{\overline{C_1}}) \to \mathbb{1}_{C_0}$$
and hence sits in the exact sequence
$$0 \to \mathcal{IC}(\mathbb{1}_{C_0}) \to \mathcal{F}_4 \to \mathcal{IC}(\mathbb{1}_{C_1}) \oplus \mathcal{IC}(\mathbb{1}_{C_2}) \to 0,$$
which we use to compute the functor Ev on these perverse sheaves in Table 4.7.3.
- On P3 (for $n = 2$) we define $\mathcal{F}_5 = \mathbb{1}_{C_3}$ which is cohomologically equivalent to a complex
$$\mathbb{1}_{\overline{C_3}}[2] \to (\mathbb{1}_{\overline{C_2}} \oplus \mathbb{1}_{\overline{C_1}})[1] \to \mathbb{1}_{C_0}[0]$$
and hence sits in the exact sequence
$$0 \to \mathcal{F}_4 \to \mathcal{F}_5 \to \mathcal{IC}(\mathbb{1}_{C_3}) \to 0,$$
which we use to compute the functor Ev on these perverse sheaves in Table 4.7.3.

4.7. **Proof of Theorem 4.2.** Let $H \times V \to V$ be one of the prehomogeneous vector spaces appearing in Proposition 1.2. The result is immediate if $V = V^s$ and $H = H^s$ or in the case P0 so we let $H^s \times V^s \to V^s$ be one of the proper prehomogeneous vector spaces appearing in Lemma 4.3. Let C' be an H^s-orbit in V^s and let $C \subseteq V$ be the saturation of C'. In Proposition 4.4 we show that if C' is V-conormal then the outside of the following diagram commutes.



It is important to note that the square in this diagram does not commute.

Proposition 4.4. *Let $H \times V \to V$ be one of the prehomogeneous vector spaces appearing in Proposition 1.2 and let $H^s \times V^s \to V^s$ be one of the prehomogeneous vector spaces appearing in Lemma 4.3. Let C' be an H^s-orbit in V^s. If C' is V-conormal then*

$$\mathrm{trace}_s\left(\mathsf{NEvs}_C[\dim C]\ \mathcal{P}\right) = \mathrm{trace}_s\left(\mathsf{NEvs}_{C'}[\dim C']\mathrm{res}\,\mathcal{P}|_{V^s}\right), \tag{14}$$

for all equivariant perverse sheaves $\mathcal{P}$ on V, where $C \subseteq V$ is the saturation of C'.

Proof. We prove this by verifying (14) by calculating the left- and right-hand sides of (14) independently, in each of the three cases appearing in Section 4.5.

(1) Case P2(ii). The main calculations are summarized by the Table 4.7.1. The calculations are completed by comparing trace$_s$ of the sheaves in the final two columns. In this case these are always 1. We note that for each C'_i the saturation is C_i.

Table 4.7.1. Summary of Calculations for Case P2(ii)

$\mathcal{P}$	res $\mathcal{P}$	C'	$\mathsf{NEvs}_C[\dim C]\ \mathcal{P}$	$\mathsf{NEvs}_{C'}[\dim C']$ res $\mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathcal{IC}(\mathbb{1}_{C'_0})$	C'_0	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_0}}[0]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_0}}[0]$
		C'_1	$0_{\Lambda^{\mathrm{sreg}}_{C_1}}[0]$	$0_{\Lambda^{\mathrm{sreg}}_{C'_1}}[0]$
$\mathcal{IC}(\mathbb{1}_{C_1})$	$\mathcal{IC}(\mathbb{1}_{C'_1})[1]$	C'_0	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_0}}[0]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_0}}[0]$
		C'_1	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_1}}[2]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_1}}[2]$

(2) Case P4(iv). This is the fixed-point prehomogeneous vector subspace $H^s \times V^s \to V^s$ for $s = \mathrm{diag}(-1, 1) \in \mathrm{GL}_2$ and $H = \mathrm{GL}_2$ acting on $V = \mathbb{A}^4$ by $\det^{-1} \otimes \mathrm{Sym}^3$; then $H^s = \mathrm{GL}_1^2$ and $V^s = \mathbb{A}^2$ with action equivalent to P3 in the case $n = 2$. Recall that we write C'_0 for the trivial H^s-orbit, C'_1 for the H^s-orbit of $(1, 0, 0, 0) \in V^s$, C'_2 for the H^s-orbit of $(0, 0, 1, 0) \in V^s$ and C'_3 for the open H^s-orbit in V^s. The equivariant fundamental group of each H^s-orbit is trivial except $A_{C'_3} = S_2$; let ρ be the non-trivial quadratic character of this group and let $\rho_{C'_3}$ be the corresponding local system on C'_3.

The main calculations are summarized by the Table 4.7.2. The calculations are completed by comparing trace$_s$ of the sheaves in the final two columns and accounting for shifts. Because the image of s in A_C^{ABV} is order 2 we have $\mathrm{trace}_s\,\varrho = 0$, $\mathrm{trace}_s\,\varepsilon = -1$, $\mathrm{trace}_s\,\rho = -1$ and $\mathrm{trace}_s\,\vartheta_2 = -1$. We note that for each C'_i the saturation is C_i.

(3) Case P4(v). The fixed-point prehomogenous space $H^s \times V^s \to V^s$ is equivalent to P3 in the case $n = 3$. Without loss of generality, we take $s = \mathrm{diag}(\theta_3^2, \theta_3)$. Recall that in Case P4(v) there are only two H^s-orbits $C' \subset V^s$ that are V-conormal: the open orbit C'_0 and the closed orbit C'_3.

The main calculations are summarized by Table 4.7.3. The calculations are completed by comparing trace$_s$ of the sheaves in the final two columns and accounting for shifts.

TABLE 4.7.2. Summary of Calculations for Case P4(iv). In this table we write $\vartheta : \{1, \theta_2\} \to \mathbb{C}^\times$ for the character generally denoted in this paper by ϑ_2, defined by $\vartheta(\theta_2) = \theta_2$.

$\mathcal{P}$	$\operatorname{res} \mathcal{P}$	C'	$\mathsf{NEvs}_C[\dim C]\ \mathcal{P}$	$\mathsf{NEvs}_{C'}[\dim C']\ \operatorname{res} \mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathcal{IC}(\mathbb{1}_{C'_0})$	C'_0	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C_0}}[0]$	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C'_0}}[0]$
		C'_1	0	0
		C'_2	0	0
		C'_3	0	0
$\mathcal{IC}(\mathbb{1}_{C_1})$	$\mathcal{IC}(\mathbb{1}_{C'_1})[1]$	C'_0	$\varrho_{\Lambda^{\text{sreg}}_{C_0}}[0]$	0
		C'_1	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C_1}}[2]$	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C'_1}}[2]$
		C'_2	0	0
		C'_3	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	$\mathcal{F}_2[2] \oplus \mathcal{IC}(\mathbb{1}_{C'_0})[1]$	C'_0	0	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C'_0}}[2] \oplus \mathbb{1}_{\Lambda^{\text{sreg}}_{C'_0}}[1]$
		C'_1	$\vartheta_{\Lambda^{\text{sreg}}_{C_1}}[2]$	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C'_1}}[3]$
		C'_2	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C_2}}[3]$	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C'_2}}[3]$
		C'_3	0	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	$\mathcal{IC}(\mathbb{1}_{C'_3})[2]$	C'_0	0	0
		C'_1	0	0
		C'_2	0	0
		C'_3	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C_3}}[4]$	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C'_3}}[4]$
$\mathcal{IC}(\varrho_{C_3})$	$(\mathcal{F}_3 \oplus \mathcal{IC}(\mathbb{1}_{C'_0}) \oplus \mathcal{IC}(\rho_{C'_3}))[2]$	C'_0	0	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C'_0}}[2] \oplus \vartheta_{\Lambda^{\text{sreg}}_{C'_0}}[2]$
		C'_1	0	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C'_1}}[3] \oplus \vartheta_{\Lambda^{\text{sreg}}_{C'_1}}[3]$
		C'_2	$\vartheta_{\Lambda^{\text{sreg}}_{C_2}}[3]$	$\vartheta_{\Lambda^{\text{sreg}}_{C'_2}}[3]$
		C'_3	$\varrho_{\Lambda^{\text{sreg}}_{C_3}}[4]$	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C'_3}}[4] \oplus \vartheta_{\Lambda^{\text{sreg}}_{C'_3}}[4]$
$\mathcal{IC}(\varepsilon_{C_3})$	$\mathcal{IC}(\rho_{C'_3})[2]$	C'_0	$\varepsilon_{\Lambda^{\text{sreg}}_{C_0}}[0]$	$\vartheta_{\Lambda^{\text{sreg}}_{C'_0}}[2]$
		C'_1	$\mathbb{1}_{\Lambda^{\text{sreg}}_{C_1}}[2]$	$\vartheta_{\Lambda^{\text{sreg}}_{C'_1}}[3]$
		C'_2	$\vartheta_{\Lambda^{\text{sreg}}_{C_2}}[3]$	$\vartheta_{\Lambda^{\text{sreg}}_{C'_2}}[3]$
		C'_3	$\varepsilon_{\Lambda^{\text{sreg}}_{C_3}}[4]$	$\vartheta_{\Lambda^{\text{sreg}}_{C'_3}}[4]$

Because the image of s in A^{ABV}_C has order 3 (or order 1) we have $\operatorname{trace}_s \vartheta_2 = 1$, $\operatorname{trace}_s \varepsilon = 1$, $\operatorname{trace}_s \varrho = \theta_3 + \theta_3^2 = -1$ and $\operatorname{trace}_s \rho \oplus \rho^2 = \theta_3 + \theta_3^2 = -1$. Note that saturation of C'_1 and C'_2 are both C_1, these do not satisfy the hypotheses, and that these cases do not always satisfy (14).

□

4.8. **Geometric lifting of stable distributions.**

From Section 3, recall Definition 5:

$$\Theta^{G^\delta}_{\phi,s} := e(\delta) \sum_{\pi \in \Pi^{\text{ABV}}_\phi(G^\delta(F))} \operatorname{trace}_s \left(\mathsf{NEvs}_\phi[\dim(\phi)]\mathcal{P}(\pi)[-\dim(\pi)]\right)\ \Theta_\pi,$$

TABLE 4.7.3. Summary of Calculations for Case P4(v). In this table we write $\vartheta : \{1, \theta_3, \theta_3^2\} \to \mathbb{C}^\times$ for the character generally denoted in this paper by ϑ_3, defined by $\vartheta(\theta_3) = \theta_3$.

$\mathcal{P}$	res $\mathcal{P}$	C'	$\mathsf{NEvs}_C[\dim C]\ \mathcal{P}$	$\mathsf{NEvs}_{C'}[\dim C']$ res $\mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathcal{IC}(\mathbb{1}_{C'_0})$	C'_0	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_0}}[0]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_0}}[0]$
		C'_1	0	0
		C'_2	0	0
		C'_3	0	0
$\mathcal{IC}(\mathbb{1}_{C_1})$	$\mathcal{F}_4[1]$	C'_0	$\varrho_{\Lambda^{\mathrm{sreg}}_{C_0}}[0]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_0}}[1]$
		C'_1	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_1}}[2]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_1}}[2]$
		C'_2	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_1}}[2]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_2}}[2]$
		C'_3	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	$\mathcal{F}_4[2] \oplus \mathcal{IC}(\mathbb{1}_{C'_0})[1]$	C'_0	0	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_0}}[2] \oplus \mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_0}}[1]$
		C'_1	$\vartheta_{\Lambda^{\mathrm{sreg}}_{C_1}}[2]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_1}}[3]$
		C'_2	$\vartheta_{\Lambda^{\mathrm{sreg}}_{C_1}}[2]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_1}}[3]$
		C'_3	0	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	$\mathcal{IC}(\mathbb{1}_{C'_3})[2]$	C'_0	0	0
		C'_1	0	0
		C'_2	0	0
		C'_3	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_3}}[4]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_3}}[4]$
$\mathcal{IC}(\varrho_{C_3})$	$(\mathcal{IC}(\mathbb{1}_{C'_0}) \oplus \mathcal{IC}(\vartheta_{C'_3}) \oplus \mathcal{IC}(\vartheta^2_{C'_3}))[2]$	C'_0	0	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_0}}[2] \oplus \vartheta_{\Lambda^{\mathrm{sreg}}_{C'_3}}[2] \oplus \vartheta^2_{\Lambda^{\mathrm{sreg}}_{C'_3}}[2]$
		C'_1	0	$\vartheta_{\Lambda^{\mathrm{sreg}}_{C'_3}}[3] \oplus \vartheta^2_{\Lambda^{\mathrm{sreg}}_{C'_3}}[3]$
		C'_2	0	$\vartheta_{\Lambda^{\mathrm{sreg}}_{C'_3}}[3] \oplus \vartheta^2_{\Lambda^{\mathrm{sreg}}_{C'_3}}[3]$
		C'_3	$\varrho_{\Lambda^{\mathrm{sreg}}_{C_3}}[4]$	$\vartheta_{\Lambda^{\mathrm{sreg}}_{C'_3}}[4] \oplus \vartheta^2_{\Lambda^{\mathrm{sreg}}_{C'_3}}[4]$
$\mathcal{IC}(\varepsilon_{C_3})$	$\mathcal{F}_5[2]$	C'_0	$\varepsilon_{\Lambda^{\mathrm{sreg}}_{C_0}}[0]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_0}}[2]$
		C'_1	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_1}}[2]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_1}}[3]$
		C'_2	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C_1}}[2]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_2}}[3]$
		C'_3	$\varepsilon_{\Lambda^{\mathrm{sreg}}_{C_3}}[4]$	$\mathbb{1}_{\Lambda^{\mathrm{sreg}}_{C'_3}}[4]$

where (G, s, ξ) is an endoscopic triple for G_2 and $\delta \in Z^1(G, F)$ is a pure inner form of G. Recall also that this distribution can be written in the form

$$\Theta^{G^\delta}_{\phi,s} := e(\delta) \sum_{\pi \in \Pi^{\mathrm{ABV}}_\phi(G^\delta(F))} (-1)^{\dim(\phi)-\dim(\pi)} \langle s, \pi\rangle\ \Theta_\pi.$$

Remark 4.5. If the Langlands parameter $\phi : W'_F \to {}^LG$ is not relevant to $G(F)$, in the sense of [Bor79], then $\Theta^{G^\delta}_{\phi,s}$ may be trivial; for example, $\Theta^{\mathrm{PGL}_3^\delta}_{\phi_{3.5.0}} = 0$ for non-trivial δ. On the other hand, $\Theta^{G^\delta}_{\phi,s}$ can be non-trivial even when $\phi : W'_F \to {}^LG$ is not relevant to $G(F)$; for example, although $\phi_{3.5.3a}$ is not relevant to PGL_3^δ for non-trivial δ, $\Theta^{\mathrm{PGL}_3^\delta}_{\phi_{3.5.3a},s} = e(\delta)\Theta_{\chi_{\mathrm{PGL}_3^\delta(F)}}$, which is non-trivial.

Definition 8. Let (G,s,ξ) be an endoscopic triple for G_2. Let $\phi : W'_F \to {}^LG$ be an unramified Langlands parameter. We define

$$\operatorname{Lift}_{(G,s,\xi)} \Theta^G_\phi := \sum_{\pi\in\Pi^{\text{ABV}}_{\xi\circ\phi}(G_2(F))} \operatorname{trace}_s\left(\mathsf{NEvs}_\phi[\dim(\phi)]\ \operatorname{res}\mathcal{P}(\pi)[-\dim(\pi)]\right)\ \Theta_\pi,$$

where res : $D_{Z_{\widehat{G_2}}(\xi\circ\phi)}(V_{\xi\circ\lambda}) \to D_{Z_{\widehat{G}}(\phi)}(V_\lambda)$ is the equivariant functor defined by $\operatorname{res}\mathcal{F} = \mathcal{F}|_{V_\lambda}$. If the context make it clear what the endoscopic triple is, we will sometimes use the abbreviated notation $\operatorname{Lift}^{G_2}_G$ for $\operatorname{Lift}_{(G,s,\xi)}$. Let $\delta \in Z^1(F,G)$ be a pure inner form of G. Then $(G,1,\mathrm{id})$ is an endoscopic triple for G^δ over F and, extending the definition above,

$$\operatorname{Lift}^{G^\delta}_G \Theta^G_\phi := e(\delta) \sum_{\pi\in\Pi^{\text{ABV}}_{\xi\circ\phi}(G^\delta(F))} \operatorname{trace}_1\left(\mathsf{NEvs}_\phi[\dim(\phi)]\ \operatorname{res}\mathcal{P}(\pi)[-\dim(\pi)]\right)\ \Theta_\pi.$$

Conjecture 1. Let (G,s,ξ) be an endoscopic triple for G_2 and let $\phi : W'_F \to {}^LG$ be a unramified Langlands parameter that is ξ-conormal. The Langlands-Shelstad transfer of Θ^G_ϕ from $G(F)$ to $G_2(F)$ is $\Theta^{G_2}_{\xi\circ\phi,s}$. Now let $\delta \in Z^1(F,G)$ be a pure inner form for G over F and suppose $\phi : W'_F \to {}^LG$ is also relevant to $G^\delta(F)$. Then the Jacquet-Langlands transfer of Θ^G_ϕ from $G(F)$ to $G^\delta(F)$ is $e(\delta)\Theta^{G^\delta}_\phi$

Theorem 4.6. *Let (G,s,ξ) be an endoscopic triple for G_2. Let $\phi : W'_F \to {}^LG$ be a unramified Langlands parameter that is ξ-conormal. Then*

$$\operatorname{Lift}^G_{(G,s,\xi)} \Theta^G_\phi = \Theta_{\xi\circ\phi,s}.$$

Let $\delta \in Z^1(F,G)$ be a pure inner form of G and suppose also that ϕ is relevant to G^δ. Then

$$\operatorname{Lift}^{G^\delta}_G \Theta^G_\phi = e(\delta)\Theta^{G^\delta}_\phi.$$

Proof. Theorem 4.2, we have

$$\operatorname{trace}_s\left(\mathsf{NEvs}_\phi[\dim(\phi)]\ \mathcal{P}\right) = \operatorname{trace}_s\left(\mathsf{NEvs}_{\xi\circ\phi}[\dim(\xi\circ\phi)]\operatorname{res}\mathcal{P}\right), \tag{15}$$

for every simple object $\mathcal{P}$ in $\mathsf{Per}_{Z_{\widehat{G_2}}(\xi\circ\lambda)}(V_{\xi\circ\lambda})$. Thus,

$$\begin{aligned}\operatorname{Lift}_{(G,s,\xi)} \Theta^G_\phi &:= \sum_{\pi\in\Pi^{\text{ABV}}_{\xi\circ\phi}(G_2(F))} \operatorname{trace}_s\left(\mathsf{NEvs}_\phi[\dim(\phi)]\operatorname{res}\mathcal{P}(\pi)[-\dim(\pi)]\right)\ \Theta_\pi \\ &= \sum_{\pi\in\Pi^{\text{ABV}}_{\xi\circ\phi}(G_2(F))} \operatorname{trace}_s\left(\mathsf{NEvs}_{\xi\circ\phi}[\dim(\xi\circ\phi)]\mathcal{P}(\pi)[-\dim(\pi)]\right)\ \Theta_\pi \\ &=: \Theta_{\xi\circ\phi,s}.\end{aligned}$$

Directly from the definitions we see

$$\begin{aligned}\operatorname{Lift}^{G^\delta}_G \Theta^G_\phi &:= e(\delta)\textstyle\sum_{\pi\in\Pi^{\text{ABV}}_{\xi\circ\phi}(G^\delta(F))} \operatorname{trace}_1\left(\mathsf{NEvs}_\phi[\dim(\phi)]\ \operatorname{res}\mathcal{P}(\pi)[-\dim(\pi)]\right)\ \Theta_\pi \\ &= e(\delta)\textstyle\sum_{\pi\in\Pi^{\text{ABV}}_{\xi\circ\phi}(G^\delta(F))} \operatorname{trace}_1\left(\mathsf{NEvs}_\phi[\dim(\phi)]\ \mathcal{P}(\pi)[-\dim(\pi)]\right)\ \Theta_\pi \\ &= e(\delta)\Theta^{G^\delta}_\phi.\end{aligned}$$

□

Definition 9. Let $\phi : W'_F \to {}^LG_2$ be an unramified Langlands parameter for $G_2(F)$. If $s \in \mathcal{S}^{\text{ABV}}_\phi$ then $\phi = \xi\circ\phi^s$ for an endoscopic triple (G,s,ξ) and a Langlands parameter $\phi^s : W'_F \to {}^LG$. If ϕ has the property that ϕ^s is ξ-conormal, we say that ϕ is s-conormal. If ϕ is of Arthur type then ϕ is s-conormal for every $s \in \mathcal{S}^{\text{ABV}}_\phi$.

Theorem 4.7. *Let $\phi : W'_F \to {}^LG_2$ be an unramified Langlands parameter that is s-conormal for every $s \in \mathcal{S}^{\text{ABV}}_\phi$. If π is any unipotent representation of $G_2(F)$ then the distribution character Θ_π may be expressed as a linear combination of the distributions $\operatorname{Lift}^{G_2}_{(G,s,\xi)} \Theta^G_{\phi^s}$, letting ϕ range* *over Langlands parameters with the same infinitesimal parameter as* π *and letting* s *range over* $\mathcal{S}_\phi^{\text{ABV}}$. *And if* $\Pi_\phi^{\text{ABV}}(G_2(F)) \to \widehat{A_\phi^{\text{ABV}}}$ *is a bijection then,*

$$\Theta_\pi = \sum_{(G,s,\xi)} (-1)^{\dim(\phi^s)-\dim(\pi)} \frac{\overline{\langle s,\pi\rangle}}{|Z_{A_\phi}(s)|}\ \mathrm{Lift}_{(G,s,\xi)}\,\Theta^G_{\phi^s}, \qquad \forall \pi \in \Pi_\phi^{\text{ABV}}(G_2(F)),$$

where the sum is taken over equivalence classes of endoscopic triples (G,s,ξ) *with* $s \in \mathcal{S}_\phi^{\text{ABV}}$ *and where we identify* s *with its image under* $\mathcal{S}_\phi^{\text{ABV}} \to A_\phi^{\text{ABV}}$ *in the calculation of* $Z_{A_\phi}(s)$.

Proof. Distribution characters for equivalence classes of irreducible admissible representations of $G_2(F)$ are linearly independent by [BZ76]. For each unramified Langlands parameter ϕ for $G_2(F)$, the number of equivalence classes of endoscopic triples for (G,s,ξ) for $G_2(F)$ with $s \in \mathcal{S}_\phi^{\text{ABV}}$ is equal to the number of irreducible representations of A_ϕ^{ABV}. This makes it possible to determine the characters $\langle s,\pi\rangle$ appearing in the sum $\Theta_{\phi,s} = \sum_\pi (-1)^{\dim(\phi)-\dim(\pi)}\langle s,\pi\rangle\ \Theta_\pi$. The rest of the theorem is a direct consequence of (Basis). □

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